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The Decision Problem for a Three-sorted Fragment of Set Theory with Restricted Quantification and Finite Enumerations ⋆

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Abstract

We solve the satisfiability problem for a three-sorted fragment of set theory (denoted $3\mathrm{LQST}_0^{\mathrm{R}}$), which admits a restricted form of quantification over individual and set variables and the finite enumeration operator $\{\text{-},\text{-},\ldots,\text{-}\}$ over individual variables, by showing that it enjoys a small model property, i.e., any satisfiable formula ψ of $3\mathrm{LQST}_0^{\mathrm{R}}$ has a finite model whose size depends solely on the length of ψ itself. Several set-theoretic constructs are expressible by $3\mathrm{LQST}_0^{\mathrm{R}}$-formulae, such as some variants of the power set operator and the unordered Cartesian product. In particular, concerning the latter construct, we show that when finite enumerations are allowed, the resulting formula is exponentially shorter than in their absence.

Keywords: Satisfiability problem, set theory, restricted quantification, finite enumerations.

1 Introduction

Computable set theory studies the decidability problem for specific collections of set-theoretic formulae (also called *syllogistics*). The main results in computable set theory up to 2001 have been collected in [7,13]. We also mention that the most efficient decision procedures for fragments of set theory form the inferential core of the proof verifier ÆtnaNova [17].

In this paper we present a decidability result for the satisfiability problem of the set-theoretic language $3\mathrm{LQST}_0^{\mathrm{R}}$ (Three-Level Quantified Syllogistic with finite enumeraTions and Restricted quantifiers), which is a three-sorted quantified syllogistic involving *individual variables*, *set variables*, and *collection variables*, ranging

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over the elements of a given nonempty universe D, the subsets of D, and the collections of subsets of D, respectively. The language of $\mathrm{3LQST}_0^{\mathrm{R}}$ admits the predicate symbols $=$ and $\in$ and a restricted form of quantification over individual and set variables. The language $\mathrm{3LQST}_0^{\mathrm{R}}$ extends $\mathrm{3LQS}^{\mathrm{R}}$ presented in [9] as it admits the finite enumeration operator $\{\text{-},\text{-},\ldots,\text{-}\}$ over individual variables. In spite of its simplicity, $\mathrm{3LQST}_0^{\mathrm{R}}$ allows one to express several constructs of set theory. Among them, the most comprehensive one is the set former, which in turn allows one to express other set-theoretic operators like some variants of the power set and the unordered Cartesian product. Concerning the latter, we will see that it can be expressed by $\mathrm{3LQST}_0^{\mathrm{R}}$-formulae of linear length. On the other hand, if the finite enumeration operator is dropped, exponentially long $\mathrm{3LQS}^{\mathrm{R}}$-formulae are required to express it.

Much as for $\mathrm{3LQS}^{\mathrm{R}}$, we will show that the fragment $\mathrm{3LQST}_0^{\mathrm{R}}$ enjoys a small model property. The proof is carried out by showing how to extract, out of a given model satisfying a $\mathrm{3LQST}_0^{\mathrm{R}}$-formula ψ, another model of ψ but of bounded finite cardinality.

The paper is organized as follows. In Section 2 we introduce some related work in computable set theory concerning multi-sorted stratified syllogistics. Then, in Section 3, we first present the syntax and semantics of a more general language, denoted $\mathrm{3LQST}_0$, and then provide a decidable semantic restriction to characterize the fragment $\mathrm{3LQST}_0^{\mathrm{R}}$ of our interest. Subsequently, in Section 4, we show that several set-theoretic constructs are readily expressible by $\mathrm{3LQST}_0^{\mathrm{R}}$-formulae. In Section 5, the machinery needed to prove our main decidability result is provided and, in Section 6, the small model property for $\mathrm{3LQST}_0^{\mathrm{R}}$ is sketched, thus solving the satisfiability problem for $\mathrm{3LQST}_0^{\mathrm{R}}$. Then, in Section 7 we present two distinct representations of the unordered Cartesian product. The first one, using the finite enumeration operator, is linear in the length of the product, the second one, not involving the finite enumeration operator, is exponentially longer. Finally, in Section 8, we draw our conclusions.

2 Related work

Most of the decidability results established in computable set theory concern one-sorted multi-level syllogistics, namely collections of formulae involving variables of one type only, ranging over the von Neumann universe of sets. On the other hand, few decidability results have been proved for multi-sorted stratified syllogistics, admitting variables of several types. This, despite of the fact that in many fields of computer science and mathematics often one deals with multi-sorted languages.

An efficient decision procedure for the satisfiability of the Two-Level Syllogistic language (2LS), a fragment admitting variables of two sorts (for individuals and for sets of individuals), the basic set-theoretic operators such as $\cup$, $\cap$, $\setminus$, the relators $=$, $\in$, $\subseteq$, and propositional connectives, has been presented in [15]. The three-sorted language 3LSSPU (Three-Level Syllogistic with Singleton, Powerset, and general Union), allowing three types of variables, and the singleton, powerset, and general union operators, in addition to the operators and predicates already in 2LS, has