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Electronic Notes in Theoretical Computer Science 322 (2016) 103–118

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Is Hyper-extensionality Preservable Under Deletions of Graph Elements? [1]

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Abstract

Any hereditarily finite set S can be represented as a finite pointed graph –dubbed membership graph– whose nodes denote elements of the transitive closure of $\{S\}$ and whose edges model the membership relation. Membership graphs must be hyper-extensional, that is pairwise distinct nodes are not bisimilar and (uniquely) represent hereditarily finite sets.

We will see that the removal of even a single node or edge from a membership graph can cause "collapses" of different nodes and, therefore, the loss of hyper-extensionality of the graph itself. With the intent of gaining a deeper understanding on the class of hyper-extensional hereditarily finite sets, this paper investigates whether pointed hyper-extensional graphs always contain either a node or an edge whose removal does not disrupt the hyper-extensionality property.

Keywords: Set theory, Hereditarily finite sets, Non-well-foundedness, Hyper-extensionality.

1 Introduction

A set is hereditarily finite if it is finite and all its elements are hereditarily finite. Moreover, it is well-founded if any chain of membership relations starting from it is finite. In standard Set Theory the Extensionality axiom, establishing that two sets

[1] This work has been partially supported by GNCS-INDAM project "*Algoritmica per il model checking e la sintesi di sistemi safety-critical*" and by University of Trieste FRA project "*Learning specifications and robustness in signal analysis (with a case study related to health care)*".

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http://dx.doi.org/10.1016/j.entcs.2016.03.008

are equal if and only if they have the same elements, guarantees that hereditarily finite well-founded sets can be inductively constructed starting from the empty set.

When also cyclic chains of memberships are allowed sets are called non-well-founded and one of the possible principles for establishing equality is Aczel's Anti-Foundation axiom based on the notion of bisimulation [2].

A hereditarily finite set S can be canonically represented through a pointed finite graph G in which each node represents a different element of the transitive closure of $\{S\}$ and the edges of G model the membership relation. Since the notion of bisimulation can be naturally defined also on graphs, this means that in the canonical representation of S there are not two different bisimilar nodes. Well-founded sets are represented by acyclic graphs, while non-well-founded sets are represented by cyclic ones (e.g., see [2] for more details).

Now a quite general question arises: is there a natural way to inductively reason on both well-founded and non-well-founded hereditarily finite sets represented through graphs? In other terms, is there a way to inductively construct/deconstruct graphs representing hereditarily finite sets? Such question has been previously formalized and studied in [14] where the authors ask whether given the canonical representation of a set, it is always possible to find a node which can be removed producing the canonical representation of another set, i.e., without causing any bisimulation collapse. A definitive answer is not provided in [14]. In this paper we further investigate in that direction proving that there are cases in which it is not possible to remove any node without causing collapses. On the other hand, we provide positive evidence on the fact that there always exists an edge which can be safely removed. This result is achieved by introducing the notion of *n-well-founded part* of a non-well-founded graph and by applying Ackermann code on it.

The paper is organized as follows: Section 2 formalizes hereditarily finite sets. Section 3 relates hereditarily finite sets and pointed hyper-extensional graphs and defines keystones –elements whose removal disrupts the graph hyper-extensionality. Section 4 presents a pipeline to enumerate pointed hyper-extensional graphs. This pipeline is used in Section 5 to prove that there exist pointed hyper-extensional graphs whose nodes (edges) are all keystones. Section 6 introduces the notion of disposable element –an element whose removal does not produce collapses between nodes of the same connected components– and shows a pointed hyper-extensional graph that do not contain disposable nodes. In Section 7, we prove that pointed hyper-extensional graphs always have a disposable edge. Finally, in Section 8, we draw conclusions and suggest future works.

2 Hereditarily Finite Sets

Hereditarily finite sets are finite sets whose elements are hereditarily finite sets.

We write $\mathcal{P}(S)$ to denote the powerset of S i.e. $\mathcal{P}(S) = \{S' \mid S' \subseteq S\}$

Definition 2.1 [Well-founded Hereditarily Finite Sets] *Well-founded hereditarily finite sets* are the elements of $\mathrm{HF} \stackrel{\mathrm{def}}{=} \bigcup_{i \in \mathbb{N}} \mathrm{HF}_i$ where the HF_i's are defined as