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Multi-Linear Iterative K-Σ-Semialgebras

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Abstract

We consider K-semialgebras for a commutative semiring K that are at the same time Σ-algebras and satisfy certain linearity conditions. When each finite system of guarded polynomial fixed point equations has a unique solution over such an algebra, then we call it an iterative multi-linear K-Σ-semialgebra. Examples of such algebras include the algebras of Σ-tree series over an alphabet A with coefficients in K, and the algebra of all rational tree series. We show that for many commutative semirings K, the rational Σ-tree series over A with coefficients in K form the free multi-linear iterative K-Σ-semialgebra on A.

Keywords: Semialgebra, Σ-algebra, rational tree-series, free algebra, unique fixed points.

1 Introduction

In an iterative algebra [1,9,23,20], certain finite systems of fixed-point equations have unique solutions. It was first pointed out in [20] that iterative algebras form a quasi-variety, hence all free iterative algebras exist. Clearly, the same holds for the subclass of iterative algebras in any variety or quasi-variety of algebras. However, it is usually a nontrivial task to find a concrete description of the free algebras in a certain class of iterative algebras.

Free algebras in the class of all iterative algebras may be represented as algebras of regular (finite or infinite) trees, cf. [23]. A characterization of the free iterative idempotent semirings by regular word languages is implicit in Salomaa's axiomatization of regular languages [21]. Extensions of Salomaa's axiomatization to rational power series (or rational weighted languages) are given for fields in [19], for commutative rings in [16], and for commutative "proper semirings" (including

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all finite commutative semirings and commutative rings) in [12]. It follows from these results that in certain classes of iterative semirings or iterative semialgebras, the free algebras may be represented as algebras of rational power series. (Some results along this line are claimed in the papers [4,24], but the proofs are incomplete, as pointed out in the MR review of the second paper.) As an important consequence of the concrete representation, one can derive several decidability and undecidability properties of the corresponding equational theories and use automata theoretic arguments to establish the validity of equations. For example, it follows that the equational theory of the iterative algebras of [23,20] is decidable in polynomial time, whereas the equational theory of iterative idempotent semirings is PSPACE-complete, cf. [22].

The aim of the present paper is to extend some results from [12] to "iterative multi-linear K-Σ-algebras", which are both Σ-algebras and K-semialgebras for a fixed commutative semiring K, satisfying certain natural linearity conditions. The main result shows that when K is proper, the free iterative multi-linear K-Σ-algebras may be described by rational tree series [8]. The result applies to the semiring $\mathbb{N}$ of natural numbers, all finite commutative semirings, all commutative rings, all Noetherian commutative semirings and in fact to all commutative semirings K such that every finitely generated subsemiring of K is contained in a Noetherian subsemiring.

The unique fixed point rule has been the basis of several complete axiomatization results in Computer Science. In addition to automata and languages, it has frequently been used in concurrency, see [17,18,5], for example. Our freeness result yields a family of such completeness results for rational tree series (regular weighted tree languages) [8,11].

2 Iterative multi-linear algebras

A *semiring* $S = (S, +, \cdot, 0, 1)$ consists of a commutative monoid $(S, +, 0)$ and a monoid $(S, \cdot, 1)$ such that product distributes over all finite sums including the empty sum, so that $x \cdot 0 = 0 = 0 \cdot x$ for all $x \in S$. Examples of semirings include all rings, the semiring $\mathbb{N}$ of natural numbers and the Boolean semiring $\mathbb{B}$ over the set $\{0, 1\}$ whose sum operation is disjunction and whose product operation is conjunction. A *commutative semiring* is a semiring whose product is commutative.

Suppose that $K = (K, +, \cdot, 0, 1)$ is a commutative semiring and Σ is a ranked set. A *multi-linear K-Σ-algebra* B is both a Σ-algebra with operations $\sigma^B : B^n \to B$, for all $\sigma \in \Sigma_n$, $n \geq 0$, and a K-semimodule $(B, +, 0)$ with left K-action $K \times B \to B$ subject to the usual laws, where $x, y, z \in B$ and $k, k' \in K$:

(1) $(x + y) + z = x + (y + z)$

(2) $x + y = y + x$

(3) $x + 0 = x$

(4) $(k + k')x = kx + k'x$

(5) $0x = 0$