Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science 249 (2009) 39–60
Electronic Notes in Theoretical Computer Science
www.elsevier.com/locate/entcs

# Monoidal Indeterminates and Categories of Possible Worlds

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**Abstract**

Given any symmetric monoidal category $\mathbf{C}$, a small symmetric monoidal category $\mathbf{\Sigma}$ and a strong monoidal functor $j: \mathbf{\Sigma} \longrightarrow \mathbf{C}$, it is shown how to construct $\mathbf{C}[\boldsymbol{x}: j\mathbf{\Sigma}]$, a *polynomial* such category, the result of freely adjoining to $\mathbf{C}$ a system $\boldsymbol{x}$ of monoidal indeterminates for every object $j(w)$ with $w \in \mathbf{\Sigma}$ satisfying a naturality constraint with the arrows of $\mathbf{\Sigma}$. As a special case, we show how to construct the free co-affine category (symmetric monoidal category with initial unit) on a given small symmetric monoidal category. It is then shown that all the known categories of "possible worlds" used to treat languages that allow for dynamic creation of "new" variables, locations, or names are in fact instances of this construction and hence have appropriate universality properties.

*Keywords:* indeterminates, symmetric monoidal categories, possible-world semantics, universality

## 1 Introduction

The concept of a *polynomial algebra* $R[x]$, constructed from an algebra $R$ by freely adjoining an *indeterminate* element $x$, is familiar from algebra. Similarly, Lambek and Scott [12, Part I, Section 5] show how to construct a cartesian (or cartesian closed) polynomial category $\mathbf{C}[x: c]$ from a base cartesian (closed) category $\mathbf{C}$ by freely adjoining an indeterminate arrow $x: 1 \longrightarrow c$.

The polynomial algebra $R[x]$ is the "most general" such extension of $R$. Similarly, the polynomial category $\mathbf{C}[x: c]$ is the most general cartesian (closed) extension of $\mathbf{C}$ containing indeterminate $x$. Such properties are proved as *universality* results. For example, consider the embedding $R_x: \mathbf{C} \longrightarrow \mathbf{C}[x]$ of $\mathbf{C}$ into $\mathbf{C}[x: c]$, any

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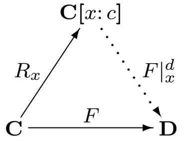
[3] This research was supported by a Discovery grant from the Natural Sciences and Engineering Research Council of Canada.


doi:10.1016/j.entcs.2009.07.083

cartesian (closed) functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$, and any $d\colon 1 \longrightarrow F(c)$ in $\mathbf{D}$; then there exists a *unique* cartesian (closed) functor $F|^d_x$ from $\mathbf{C}[x\colon c]$ to $\mathbf{D}$ such that $(F|^d_x)(x) = d$ and $F|^d_x \cdot R_x = F$:




In this work, we develop comparable technology for *symmetric monoidal* categories [14]. Given a symmetric monoidal category $\mathbf{C}$, a small symmetric monoidal category $\mathbf{\Sigma}$, and a strong monoidal functor $j\colon \mathbf{\Sigma} \longrightarrow \mathbf{C}$, we show how to construct $\mathbf{C}[\boldsymbol{x}\colon j\mathbf{\Sigma}]$, the symmetric monoidal polynomial category that results from freely adjoining, for every object $j(w)$ for $w \in \mathbf{\Sigma}$, indeterminates $x_{j(w)}\colon I \longrightarrow w$ satisfying a naturality constraint with respect to the arrows of $\mathbf{\Sigma}$. When $\mathbf{\Sigma}$ is the sub-symmetric monoidal category freely generated by some set of $\mathbf{C}$ objects [4] the indeterminates are completely "free," as in the examples described above.

We believe this construction has many applications. As our leading examples, we consider the categories of "possible worlds" that have been used in the semantics of imperative programming languages. John Reynolds and Frank Oles [31,23,24,25,19] show how block-structured storage management in Algol-like languages [22] may be explicated using a semantics based on functor categories $\mathbf{W} \Rightarrow \mathbf{S}$, where $\mathbf{W}$ is a suitable category of "worlds" characterizing local aspects of storage structure, and $\mathbf{S}$ is a conventional semantic category of sets or domains. Every programming-language type $\theta$ is interpreted as a functor $[\![\theta]\!]\colon \mathbf{W} \longrightarrow \mathbf{S}$ and every programming-language term-in-context $\pi \vdash X\colon \theta$ is interpreted as a natural transformation $[\![\pi \vdash X\colon \theta]\!]\colon [\![\pi]\!] \stackrel{\cdot}{\longrightarrow} [\![\theta]\!]$.

Oles gives two presentations of his category of worlds and shows that they are equivalent. Reynolds presents what *seems* to be a different category of worlds; however, it has recently been shown [7] that, under reasonable closedness assumptions, it is in fact equivalent to Oles's category.

The functor-category framework has also been exploited to analyze noninterference in Reynolds's specification logic [30,32,36,16,20], block expressions in Algol-like languages [35], and passivity in a variant of Reynolds's Syntactic Control of Interference [29,17]. These applications used a related but significantly different category of worlds, due to Tennent.

Several authors [15,28,33,34,3] have used *finite sets* (of locally available "locations" or "names") as worlds, with injections as the morphisms.

What is noteworthy about all of this work is that the categories of worlds involved have been developed in *ad hoc* fashion and their properties have not been well understood. We show here that all of these categories of worlds are instances of our monoidal polynomial construction and have *universality* properties.

The construction of $\mathbf{C}[\boldsymbol{x}\colon j\mathbf{\Sigma}]$ and its key properties, such as universality, and

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4 i.e., the sub-symmetric monoidal category consisting of all tensorings of the objects, with arrows being the relevant structural isomorphisms of $\mathbf{C}$.