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Categories of Timed Stochastic Relations

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Abstract

Stochastic behavior—the probabilistic evolution of a system in time—is essential to modeling the complexity of real-world systems. It enables realistic performance modeling, quality-of-service guarantees, and especially simulations for biological systems. Languages like the stochastic pi calculus have emerged as effective tools to describe and reason about systems exhibiting stochastic behavior. These languages essentially denote continuous-time stochastic processes, obtained through an operational semantics in a probabilistic transition system. In this paper we seek a more descriptive foundation for the semantics of stochastic behavior using categories and monads. We model a first-order imperative language with stochastic delay by identifying probabilistic choice and delay as separate effects, modeling each with a monad, and combining the monads to build a model for the stochastic language.

Keywords: probability, stochastic behavior, category theory, monads, partial additivity

1 Introduction

Stochastic temporal behavior is crucial for modeling real-world systems with non-functional requirements like quality-of-service guarantees [11]. Such requirements often take the form of soft real-time constraints such as "do a before time t with probability 0.99". Multimedia applications and collaborative virtual environments are well-known examples of systems exhibiting such characteristics.

To model and program systems with soft constraints, we need languages to express probability distributions over the delays experienced during the evolution of the system. PEPA [26] and the stochastic pi calculus [43] are two examples of languages that express this kind of stochastic temporal behavior. The semantics of these languages is operational, given in terms of a labelled probabilistic transition system. The transition systems themselves denote continuous-time stochastic processes, often continuous-time Markov stochastic processes.

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An operational semantics in terms of probabilistic transition systems, however, does not directly describe stochastic temporal behavior—it delegates the task to the metatheory. In this paper, we initiate a study of the *foundation* of stochastic temporal behavior, working towards a natural denotational semantics for stochastic languages. In particular, we are interested in relating the kind of probabilistic behavior found in languages with probabilistic choice with the kind of stochastic temporal behavior found in the stochastic pi calculus. Following Giry's [21] approach to the categorical foundation of discrete- and continuous-time Markov processes, we explore a categorical model of stochastic temporal behavior. This approach has two immediate advantages. First, the resulting model is sufficiently abstract to be generally applicable. Second, we obtain a principled derivation of semantic models for stochastic languages.

To ground our intuitions, we study stochastic temporal behavior in the context of a simple language of while loops [24,52]. Languages of while loops are relatively simple, yet structured and Turing-complete. Moreover, being first-order, their denotational semantics requires no heavy-duty machinery. In §2, we review the standard categorical semantics for such languages, taking advantage of Moggi's insight that monads can be used to lift the semantics of a pure language to an extension with effects [37,38]. We illustrate the approach with two different effects: iteration and probabilistic choice.

Stochastic temporal behavior ultimately amounts to adding delay to computations. In §3, we present an abstract approach for adding delay to the categorical semantics of our imperative language by introducing a monad to express timed computations. We examine how this monad interacts with monads capturing other effects in the language. In particular, we investigate conditions under which adding timed computations to a semantic model that correctly handles iteration yields an extended semantic model that also correctly handles iteration.

In §4, we instantiate our abstract approach to derive semantic models for a language of while loops extended with a deterministic delay operator and for a language of while loops extended with a probabilistic delay operator. In §5, we instantiate our abstract approach to derive semantic models for a *probabilistic* language of while loops [32] extended with a deterministic delay operation. We call the resulting semantic models categories of $\mathcal{M}$-timed stochastic relations $\mathbf{TSRel}_{\mathcal{M}}$, extending the category **SRel** of stochastic relations commonly used to give semantics to probabilistic languages of while loops. In these categories, we draw a relationship between probabilistic choice and stochastic temporal behavior by showing that both are in fact derivable from a primitive that lets us sample probability distributions.

We review related work in §6 and conclude in §7. Due to space restrictions, proofs of our technical results are only sketched where useful, and full proofs have been relegated to the full version of the paper [12].