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Elgot Theories: A New Perspective of Iteration Theories (Extended Abstract)⋆

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Abstract

The concept of iteration theory of Bloom and Ésik summarizes all equational properties that iteration has in usual applications, e.g., in Domain Theory where to every system of recursive equations the least solution is assigned. However, this assignment in Domain Theory is also functorial. Yet, functoriality is not included in the definition of iteration theory. Pity: functorial iteration theories have a particularly simple axiomatization, and most of examples of iteration theories are functorial.
The reason for excluding functoriality was the view that this property cannot be called equational. This is true from the perspective of the category **Sgn** of signatures as the base category: whereas iteration theories are monadic (thus, equationally presentable) over **Sgn**, functorial iteration theories are not. In the present paper we propose to change the perspective and work, in lieu of **Sgn**, in the category of sets in context (the presheaf category of finite sets and functions). We prove that Elgot theories, which is our name for functorial iteration theories, are monadic over sets in context. Shortly: from the new perspective functoriality is equational.

Keywords: iteration theory, Elgot theory, iterative algebra, rational monad

1 Introduction

In Domain Theory one works in a continuous theory and one uses iteration expressed by the fact that for every equation-morphism $e\colon n \longrightarrow n+k$ there exists the least solution $e^\dagger\colon n \longrightarrow k$. This dagger operation $e \longmapsto e^\dagger$ enjoys a number of equational properties, e.g., the fact that $e^\dagger$ is a solution of e is the equation $e^\dagger = [e^\dagger, \mathrm{id}_k]\cdot e$. The aim of the concept of iteration theory of Stephen Bloom and Zoltan Ésik was to collect all equational properties of the dagger operation in Domain Theory (and in a substantial number of other applications where iteration

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is used, see the fundamental monograph [12]). The function $e \longmapsto e^\dagger$ in Domain Theory is also functorial, that is, for every given k we obtain a functor $(-)^\dagger$ from the category of all equation morphisms $e\colon n \longrightarrow n + k$ to the slice category of k. This important property of functoriality is studied in various contexts, e.g., Alex Simpson and Gordon Plotkin call it parametrized uniformity in [22], and they say in their introduction that this is "a convenient tool for establishing that the equations of an iteration operator are satisfied". Larry Moss observed in [21] that functorial iteration theories allow for a particularly simple axiomatization. Functoriality is, however, *not* a part of the definition of iteration theory; this property is called "functorial dagger implication" in the monograph [12]. The name and the non-inclusion into the definition both indicate that Bloom and Ésik do not consider functoriality an equational property. The aim of the present paper is to demonstrate that from a new perspective functoriality *is* equational. Thus *Elgot theories* which is our name for functorial iteration theories, form an important class of equationally specified algebraic theories. They are, as proved by Martin Hyland and by Masahito Hasegawa [18], precisely those theories that are traced cocartesian categories where the trace operation is uniform for all base morphisms.

Recall that for every signature Σ the free continuous theory on Σ is the theory $\mathbb{T}_{\Sigma_\perp}$ of $\Sigma_\perp$-trees: one adds to Σ a new nullary symbol $\perp$, forming a new signature $\Sigma_\perp$, and the morphisms from 1 to n in $\mathbb{T}_{\Sigma_\perp}$ are all $\Sigma_\perp$-trees (finite and infinite) on n variables. As proved by Bloom and Ésik, the free iterative theory on Σ is the subtheory $\mathbb{R}_{\Sigma_\perp}$ of all *rational* $\Sigma_\perp$-trees, that is, trees with finitely many subtrees up to isomorphism. This defines a monad Rat on the category **Sgn** of signatures:

$$\mathsf{Rat}(\Sigma) = \text{the signature of rational } \Sigma_\perp\text{-trees.}$$

We have proved recently that the Eilenberg-Moore algebras for this monad Rat are precisely the iteration theories, see [6]. It then follows from a general theory of equational presentations due to Max Kelly and John Power [19], recalled briefly in the Appendix below, that iteration theories are equationally presentable over **Sgn**. And the corresponding equations for dagger are precisely those that hold in Domain Theory since they are precisely those that hold in the theories $\mathbb{T}_{\Sigma_\perp}$ or $\mathbb{R}_{\Sigma_\perp}$. In contrast, Elgot theories are not monadic over the category of signatures.

However, free iteration theories exist not only on all signatures, but also on all sets in context, as we proved in [4]. The latter means objects of the functor category $\mathbf{Set}^{\mathscr{F}}$ where $\mathscr{F}$ is the category of natural numbers and all functions between them. Thus, a set in context X assigns (like a signature) to every $n \in \mathbb{N}$ a set $X(n)$ which we can consider as the set of all "formulas of type X in n variables". And (unlike a signature) it assigns to every function $\varphi\colon n \longrightarrow m$ "changing variable names" a function $X\varphi\colon X(n) \longrightarrow X(m)$ of the corresponding "renaming of free variables" in formulas. See for example the semantics of λ-calculus presented by M. Fiore et al. [15] where λ-formulas are treated as a set in context. It follows from our results in [4] that for every set in context, $X \in \mathbf{Set}^{\mathscr{F}}$, a *rational theory* $\mathbb{R}_X$ can be constructed analogously to the rational-tree theory for a signature (see also [3] for concrete descriptions of those theories $\mathbb{R}_X$). Moreover, in [7] we proved that