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A Convenient Category of Domains

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Abstract

We motivate and define a category of *topological domains*, whose objects are certain topological spaces, generalising the usual ω-continuous dcppos of domain theory. Our category supports all the standard constructions of domain theory, including the solution of recursive domain equations. It also supports the construction of free algebras for (in)equational theories, can be used as the basis for a theory of computability, and provides a model of parametric polymorphism.

Keywords: Domain theory, denotational semantics, topological domain theory, TTE, convenient topology

Dedicated to Gordon Plotkin on the occasion of his 60th birthday

1 Introduction

A strong theme in Gordon Plotkin's work on domain theory is an emphasis on presenting domain theory as a toolkit for the semanticist. In particular, in his "Pisa" notes [38] (an early version of which bears a title that explicitly reflects this perspective [37]), he highlights the variety of different constructions that domain theory supports, motivating each by its computational relevance, and discussing in detail how they may be combined for semantic purposes. Hand-in-hand with this is a mathematical emphasis on grouping domains collectively into categories, so that the constructions on them get explained in terms of their universal properties. This emphasis presumably reflects an early awareness by Plotkin that, should traditional domains turn out not to fulfil all semantic needs, one might nevertheless expect other candidate notions of domain to provide much the same in the way of category-theoretic structure. Later, such considerations lay at the core of the development of axiomatic domain theory in the 1990's — a theory to which Plotkin himself made important contributions, see, e.g., [11].

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The motivation for the present article lies in observations by Plotkin concerning deficiencies in the semantic toolkit provided by traditional domain theory. In domain theory, it is known how to model: (i) higher-order types (using cartesian closed categories of domains); (ii) computability (using ω-continuous dcpos); and (iii) general computational effects such as nondeterminism (as free algebras for inequational theories). Furthermore, it is possible to combine any *two* of these features. (For (i)+(ii), use any of the cartesian closed full subcategories of ω-continuous dcpos; for (ii)+(iii), use the category of ω-continuous dcpos itself; and, for (i)+(iii), use the category of all dcpos.) However, Plotkin observed that it is not possible to combine all three. (None of the cartesian closed subcategories of ω-continuous dcpos are closed under the formation of free algebras.) This observation led Plotkin to ask for someone to find a category of domains that does support all three features in combination. Indeed, at the 2002 meeting in Copenhagen honouring Dana Scott's 70th birthday, Plotkin publicly expressed the wish to receive such a category of domains as a future birthday present for himself. This article is the requested present.

Actually, it was clear to anyone with detailed knowledge of the work on synthetic domain theory from the 1990's [35,22,30,42,34] that such categories of domains were achievable, as long as one was willing to allow them to arise as not easily describable subcategories of realizability toposes. However, we took the main challenge of Plotkin's wish to be to obtain such a category as close in spirit to the familiar categories of domain theory as possible. The approach presented here began with Simpson's observation that one particular category of domains arising in synthetic domain theory has a straightforward alternative description as a category of topological spaces [49,2]. The purpose of the present paper is to show that this category can be derived from first principles without any reference to synthetic domain theory. Indeed, it is obtained as the result of a certain natural combination of topological and domain-theoretic concerns.

Since the early days of domain theory, it has enjoyed a symbiotic relationship with general topology, see [15] for an overview. This is no accident. As Smyth observed, cf. [50,53], there is a strong analogy between open sets in topology and observable properties of data, according to which one should expect mathematical models of datatypes to be topological spaces. We review this connection between topology and computation in Section 2, and we use it as the starting point for our investigations.

A limitation of the analogy between topology and computation is that the mathematical world of topology contains many weird and wonderful spaces for which no connection with computation can possibly be envisaged. It is natural then to seek to explicitly identify those topological spaces that can be argued to have some plausible connection with computation. This is the task we address in Section 3. The idea is to require elements of a topological space to be representable as infinite streams of discrete data, cf. [54]. This allows a notion of *physical feasibility* to be developed, following Plotkin's related terminology in [38]. Roughly speaking, physical feasibility captures the idea that, in computation, a finite amount of output must depend only on a finite amount of input. For those topological spaces which