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Constructing differential categories and deconstructing categories of games

Jim Laird [a,∗,1], Giulio Manzonetto [b,2], Guy McCusker [a]

[a] Department of Computer Science, University of Bath, Bath, BA2 7AY, UK
[b] Univ. Paris 13, Sorbonne Paris Cité, LIPN, UMR 7030, CNRS, F-93430 Villetaneuse, France

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ABSTRACT

Differential categories were introduced by Blute, Cockett and Seely to axiomatize categorically Ehrhard and Regnier's syntactic differential operator. We present an abstract construction that takes a symmetric monoidal category and yields a differential category, and show how this construction may be applied to categories of games. In one instance, we recover the category previously used to give a fully abstract model of a nondeterministic imperative language. The construction exposes the differential structure already present in this model, and shows how the differential combinator may be encoded in the imperative language. The second instance corresponds to a new Cartesian differential category of games. We give a model of a simply-typed resource calculus, Resource PCF, in this category and show that it possesses the finite definability property. Comparison with a semantics based on Bucciarelli, Ehrhard and Manzonetto's relational model reveals that the latter also possesses this property and is fully abstract.

1. Introduction

An important aim in studying higher-order computation is to understand and control the way resources are used. One way to do this is by studying calculi designed to capture resource usage, and their denotational models. Two such calculi – the *differential* λ*-calculus* [1] of Ehrhard and Regnier and the *resource calculus* introduced by Tranquilli [2], inspired by the work of Boudol [3], are fundamentally related at the semantic level [4]: both may be interpreted using the notion of *differential category* introduced by Blute, Cockett and Seely [5]. In this paper, we study these concepts on both abstract and concrete levels. We give a construction of a differential category from any symmetric monoidal category, and use it to investigate the structure of newly discovered differential categories, relate them to existing examples, and to prove full abstraction results for *Resource PCF*, a typed programming language based on the resource calculus.

A potential source of differential categories, although not investigated hitherto, is *game semantics*: resource usage is represented rather explicitly in games and strategies. Indeed, we show that an existing games model of Idealized Algol with nondeterminism, introduced by Harmer and McCusker [6] contains a Cartesian differential operator [7], and may therefore be used to interpret Resource PCF, although this interpretation contains non-definable ("junk") finitary elements.

We then present the construction which we shall use to analyze differential categories. Its key step takes a symmetric monoidal category with countable biproducts, embeds it in its *Karoubi envelope* (idempotent splitting) and then constructs the *cofree cocommutative comonoid* on this category (by taking a sum of symmetric tensor powers) and a differential operator

∗ Corresponding author.
E-mail addresses: jiml@cs.bath.ac.uk (J. Laird), giulio.manzonetto@lipn.univ-paris13.fr (G. Manzonetto), g.a.mccusker@bath.ac.uk (G. McCusker).

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on the Kleisli category of the corresponding comonad. Since biproducts may be added to any category by free constructions, we have a way of embedding any symmetric monoidal (closed) category in a Cartesian (closed) differential category.

Although this construction is somewhat elaborate, it provides a useful tool for analyzing and relating more directly presented models. For example, applying it to the terminal (one object, one morphism) SMCC yields the key example of a differential category (and model of resource calculus [4]) based on the finite-multiset comonad on the category of sets and relations. We also show that our differential category of games embeds in one constructed from a simple symmetric monoidal category of games. By refining the strategies in these games to eliminate *history sensitive* behaviour, we obtain a constraint on strategies (∼-closure) in our directly presented model of Resource PCF which corresponds to finite definability. Another useful observation is that any functor of symmetric monoidal categories lifts to one between the differential categories constructed from them. In particular, from the terminal functor we derive a functor from our category of games and ∼-closed strategies into the relational model which is shown to be *full*. From this we may deduce that the relational model of Resource PCF is *fully abstract*.

Related works. This article is an extended version of [8]. We build and analyze differential categories as a semantic framework for Resource PCF, a programming language based on the resource calculus. The resource calculus has been recently studied from a syntactic point of view by Pagani and Tranquilli [9] for confluence results, by Manzonetto and Pagani for separability results [10] and by Pagani and Ronchi della Rocca [11] for results about solvability. This calculus is also strongly linked with the differential λ-calculus [1]; in that context Ehrhard and Regnier studied the relationships between Böhm trees, Taylor expansion and linear head reduction [12,13].

Monoidal and Cartesian differential categories have been introduced by Blute, Cockett and Seely in [5,7]. Subsequently Bucciarelli, Ehrhard and Manzonetto proposed the notion of Cartesian closed differential category, proved that such categories are sound models of the simply typed resource calculus [4] and studied a concrete example. Other examples of such categories have been given by Blute, Ehrhard and Tasson in [14] and by Carraro, Ehrhard and Salibra in [15].

In the present paper we provide a general method for turning a symmetric monoidal category into a differential category. Our construction of an exponential modality is a special case of the one proposed by Melliès, Tabareau and Tasson in [16]. Our additional contributions are the observations that, under certain circumstances, the equalizers required to build symmetric tensor powers can be obtained by splitting idempotents, and that the resulting model possesses differential structure.

Outline. In Section 2 we fix some categorical notations and recall the basic definitions concerning monoidal and Cartesian differential categories. In Section 3 we introduce Resource PCF, together with its operational and denotational semantics. Section 4 presents a direct description of Harmer and McCusker's differential category of games $\mathbf{G}^{\otimes}$. Section 5 is devoted to the categorical construction for turning any symmetric monoidal category into a differential category. In Section 6 we apply the construction to recover and analyze the category $\mathbf{G}^{\otimes}$, that will be then refined in Section 7. Finally, in Section 8 we use our construction to compare the games model with the relational model, and prove that the relational model is fully abstract for Resource PCF.

2. Differential categories

Differential categories were introduced by Blute, Cockett and Seely to formalize derivatives categorically. The authors started from monoidal categories [5], then extended the notion to Cartesian ones [7]; a further generalization to Cartesian closed categories has been made in [4] to model differential and resource λ-calculi.

Throughout this paper we will be working with categories whose hom-sets are endowed with the structure of a commutative monoid $(+, 0)$. Terminology for the various kinds of categorical structure we encounter varies widely, the adjective "additive" being particularly overloaded, so we will take care to define all our terminology as we go along.

Let us fix some notation. We write the identity map on an object A as id_A or simply A. Composition is written using infix ; in diagram order. We use $\langle f, g\rangle$ to denote the pairing of maps $f : A \to B$ and $g : A \to C$, and π_0, π_1 for the corresponding projections. In Cartesian closed category we denote the exponential object by $A \Rightarrow B$ and the curry of $f : A \times B \to C$ by $\Lambda(f) : A \to (B \Rightarrow C)$. We write $\Lambda^-(-)$ for the inverse of $\Lambda(-)$. We elide all associativity and unit isomorphisms associated with monoidal categories.

2.1. (Monoidal) differential categories

Let $\mathbf{C}$ be a commutative-monoid-enriched symmetric monoidal category: this means that it is a symmetric monoidal category, and that composition and tensor preserve the commutative monoid structure on hom-sets, so that

$$(f+g); h = f; h + g; h, \qquad k; (f+g) = k; f + k; g, \qquad f; 0 = 0 = 0; f,$$
$$(f+g)\otimes h = f \otimes h + g \otimes h, \qquad f \otimes 0 = 0.$$

A *coalgebra modality* on $\mathbf{C}$ is a comonad $(!, \delta, \epsilon)$ such that each object $!A$ is equipped with a comonoid structure

$$\Delta_A : !A \to !A \otimes !A, \qquad e_A : !A \to I.$$