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The number of convex permutominoes☆

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ABSTRACT

Permutominoes are polyominoes defined by suitable pairs of permutations. In this paper we provide a formula to count the number of convex permutominoes of given perimeter. To this aim we define the transform of a generic pair of permutations, we characterize the transform of any pair defining a convex permutomino, and we solve the counting problem in the transformed space.

1. Introduction

A polyomino (also known as lattice animal) is a finite collection of square cells of equal size arranged with coincident sides. In this paper we consider a special class of polyominoes, namely the permutominoes, that we define in a purely geometric way. Actually, the term "permutomino" arises from the fact that this object can be defined by a diagram on the plane representing a pair of permutations. Such diagrams were introduced in [8] as a tool to study Schubert varieties and used in [7] (where the term "permutaomino" appeared for the first time) and [6] in relation to Kazhdan-Lusztig R-polynomials.

Counting the number of polyominoes and permutominoes is an interesting combinatorial problem, still open in its more general form; yet, for some subclasses of polyominoes, exact formulae are known. For instance, the number of convex polyominoes (i.e., whose intersection with any vertical or horizontal line is connected) of given perimeter has been obtained in [2], whereas the enumeration problem for some subclasses of convex permutominoes has been solved in [5]. In this paper, we provide an explicit formula for the number of convex permutominoes of a given perimeter. Incidentally, we notice that an equivalent formula has been independently obtained in [4], using a totally different technique based on the ECO method.

Our counting technique is based on two basic facts. First, the boundary of every convex permutomino can be decomposed into four subpaths describing, in this order, a down/rightward, up/rightward, up/leftward, down/leftward stepwise movement. Second, for each abscissa (ordinate) there is exactly one vertical (horizontal) segment in the boundary with that coordinate. Actually, these two constraints hold not only for the boundary of convex permutominoes, but for a larger class of circuits we call admissible: in Section 3 we describe admissible circuits and we obtain their number $\mathcal{A}_n$ in Section 5. In Section 4 we characterize admissible circuits that do not define a permutomino: again we obtain their number $\mathcal{B}_n$ in Section 5. As a consequence, we get the number of convex permutominoes as the difference $\mathcal{A}_n - \mathcal{B}_n$.

2. Preliminaries

In this section, we shall recall some basic definitions and properties of polyominoes, permutominoes and generating functions.

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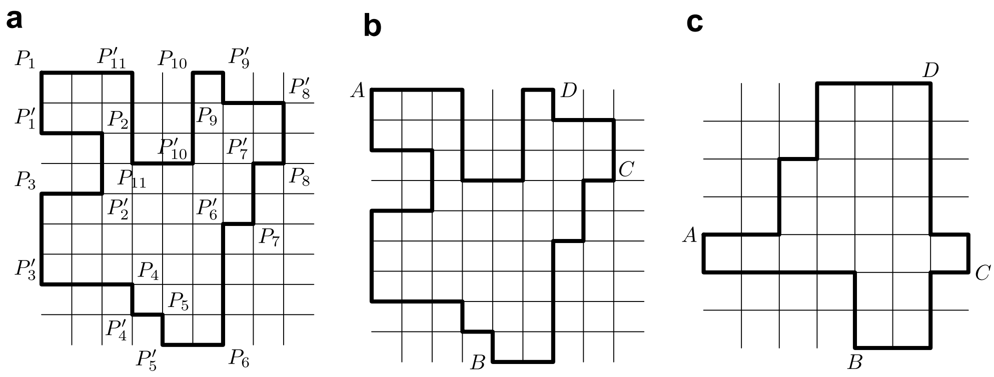

Fig. 1. (a) The boundary of a polyomino. (b) The extreme points of a polyomino. (c) The extreme points of a convex polyomino.

2.1. Polyominoes and permutominoes

A *cell* is a closed subset of $\mathbf{R}^2$ of the form $[a,a+1] \times [b,b+1]$, where $a,b \in \mathbf{Z}$; we shall identify such a cell with the pair (a,b). Let us define a binary relation $\sim$ of *adjacency* between cells by letting $(a,b) \sim (a',b')$ if and only if $a = a'$ and $|b - b'| = 1$, or $|a - a'| = 1$ and $b = b'$. A subset P of $\mathbf{R}^2$ is a *polyomino* if and only if it is a finite nonempty union of cells that is connected by adjacency, i.e., such that if $(a,b),(a',b') \in P$ then there exist $(a_1,b_1),\ldots,(a_k,b_k) \in P$ such that $(a,b) = (a_1,b_1) \sim (a_2,b_2) \sim \cdots \sim (a_k,b_k) = (a',b')$. See Fig. 1(a) for an example. A polyomino is defined up to translations; without loss of generality, we assume that the lowest leftmost vertex of the mininal bounding rectangle of the polyomino is placed at the point $(1,1)$.

Special types of polyominoes P are the following:

- P is *row-convex* if and only if $(a,b),(a',b) \in P$ and $a \le a'' \le a'$ imply $(a'',b) \in P$;
- P is *column-convex* if and only if $(a,b),(a,b') \in P$ and $b \le b'' \le b'$ imply $(a,b'') \in P$;
- P is *convex* if and only if it is both row- and column-convex;
- P is *directed* if and only if it contains at least one of the corner cells of its minimal bounding rectangle;
- P is *parallelogram* if and only if it is convex and contains at least a pair of opposite corner cells of its minimal bounding rectangle (e.g., both the lower-left and upper-right cells).

The (topological) border of a polyomino P is a disjoint union of simple closed curves; in particular, if there is only one curve, we say that P has *no holes*: all polyominoes in this work will have no holes. The border is a simple closed curve made of alternating vertical and horizontal nontrivial segments whose endpoints (*vertices*) have integral coordinates; conversely, every such a closed curve is the border of a polyomino without holes, so we shall freely identify polyominoes with their borders.

We say that P is a *permutomino of size n* if and only if its minimal bounding rectangle is a square of size $n - 1$, and the border of P has exactly one vertical segment of abscissa z and one horizontal segment of ordinate z, for every $z \in \{1,\ldots,n\}$. Notice that, since convex polyominoes have the same perimeter as their minimal bounding rectangle, a convex permutomino of size n has perimeter $4(n - 1)$.

In order to handle polyominoes we introduce the following definitions. A (stepwise) *simple path* is a sequence $P_1 = (x_1,y_1)$, $P'_1 = (x'_1,y'_1)$, $P_2 = (x_2,y_2)$, $P'_2 = (x'_2,y'_2)$ …, $P_m = (x_m,y_m)$, $P'_m = (x'_m,y'_m)$ of distinct points with integer coordinates such that, for all $i \in \{1,\ldots,m\}$, $x_i = x'_i$, and $y'_i = y_{i+1}$ if $i < m$; notice that the segments $P_iP'_i$ are vertical, whereas the segments P'_iP_{i+1} are horizontal. More generaly, a *path* is a sequence of points P_1, P'_1, …, P_k, P'_k such that, for some $m \le k$, P_1, P'_1, …, P_m, P'_m is a simple path, and for all $i > m$, $P_i = P_{i-m}$ and $P'_i = P'_{i-m}$. A *circuit* is a simple path such that $y'_m = y_1$; when dealing with circuits, we shall implicitly assume that the subscripts are treated modulo m; so, for example P_{m+1} is just P_1. A point is a (self-)*crossing point* of a simple path if and only if it is the intersection of two segments, say $P_iP'_i$ and P'_jP_{j+1}; we also say that the path has a crossing at indices (i,j).

Clearly, visiting the border of a polyomino P counter-clockwise and starting from the highest vertex of the leftmost edge, we identify a circuit without crossing points: we call it the *boundary* of P and denote it by $P_1,P'_1,P_2,P'_2,\ldots,P_m,P'_m$ (see Fig. 1(a)). Notice that if P is a permutomino, then $m = n$.

In particular we consider four special points in the boundary of any polyomino P: let $A = P_1$ be the highest vertex of the leftmost edge, B be the leftmost vertex of the lowest edge, C be the lowest vertex of the rightmost edge, D be the rightmost of the highest edge (see Fig. 1(b)). Notice that, if P is convex, then the subsequence of vertices between A and B (B and C, C and D, D and P'_m, respectively) is a path directed down/rightward (up/rightward, up/leftward, down/leftward, respectively); see Fig. 1(c).