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Computational complexity of dynamical systems: The case of cellular automata☆

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ABSTRACT

Cellular Automata can be considered discrete dynamical systems and at the same time a model of parallel computation. In this paper we investigate the connections between dynamical and computational properties of Cellular Automata. We propose a classification of Cellular Automata according to the complexities which rise from the basins of attraction of subshift attractors and investigate the intersection classes between our classification and other three topological classifications of Cellular Automata. From the intersection classes we can derive some necessary topological properties for a cellular automaton to be computationally universal according to our model.

1. Introduction

The concept of computation and Computation theory itself are strictly related to Turing Machines. In recent years, however, a new trend of investigation attempts to find connections between Dynamical System theory and Computation theory. Cellular Automata can be considered discrete dynamical systems and at the same time a model of parallel computation. It is well known that they have the same computational power of Turing Machines. There is no general agreement on the concept of universality for Cellular Automata. The universality of a cellular automaton is generally proved by showing that such automaton can simulate a universal Turing Machine [21] or some other system which is known to be computationally universal [3]. A different approach was taken by Wolfram in [23] where the author classifies empirically Cellular Automata in four classes according to the observed (by computer simulation) evolution of the automata on random configurations. He suggested that Cellular Automata in the last of his classes must be capable of universal computation. Several authors have offered formalization to Wolfram classes. We cite just few of them. Gilman [8] proposed a classification based on the concept of *equicontinuity* while Hurley [11] proposed a classification based on the concept of *attractors*. Kůrka [13] refined the Equicontinuity and Attractor classifications by using purely topological definitions and investigated the intersection classes between the two classifications and a third one based on the complexity of the languages rising from the column factors of Cellular Automata. All three classifications are based uniquely on topological concepts and it is not evident how these dynamical properties are related to computational properties of Cellular Automata except for the connection with Wolfram empirical classification.

While it is generally accepted to interpret the evolution of a dynamical system as a process of computation, it is much more less evident how to interpret the input and the output of the computation in the evolution of the system. A possible

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approach is to see the process of computation in a dynamical system as a flow toward an attractor. The attractor is considered the halting state of the computation. One such approach has been taken in [2] to develop a complexity theory for the set of continuous time dynamical systems defined by differential equations. A more general approach has been taken recently in [5]. The authors rephrase the halting problem as the problem to decide if there exists at least one configuration from some *initial set* whose orbit reaches some *halting set*. Initial and halting sets are intended to be clopen (closed and open) sets of a Cantor space so that they can be described by means of finite information. It is easy to see how these two approaches are related: in a compact metric space the orbit of some configuration converges to an attractor Z if and only if it enters into all clopen invariant sets whose omega limits coincide with Z. The authors of [5] propose a definition of universality which applies to general discrete symbolic (i.e. defined on a Cantor space) dynamical systems and they provide necessary conditions for the universality. According to their model, a universal symbolic dynamical system is not minimal (i.e. it contains at least one proper subsystem), not equicontinuos and does not satisfy the shadowing property. Moreover they conjecture that a universal dynamical system must have an infinite number of subsystems.

Here we interpret the process of computation in Cellular Automata as a flow toward a subshift attractor. A subshift attractor is an attractor which is invariant under the shift map. Subshift attractors have been investigated in [14] and [7]. We show that it is possible to restate the halting problem as the problem to decide if the omega limit of some clopen set is contained in a halting subshift attractor (that is, as the problem to decide if the orbits of all sequences contained in some clopen set converge to the attractor). We say that the computational complexity of a cellular automaton $(A^{\mathbb{Z}},F)$ with respect to the halting subshift attractor Z is defined as the complexity of clopen sets contained in the basin of attraction of Z. Since a basin of attraction is the countable union of cylinder (clopen) sets and a cylinder set can be univocally described by some word in A^*, we can characterize the complexity of basins of attraction by using Formal Language theory. We propose a classification of Cellular Automata according to the complexity of *basin languages* (Section 3). A cellular automaton with highest computational complexity has at least one subshift attractor whose basin language is recursively enumerable complete.

Since our classification is based on purely topological concepts it is easy to explore the intersection classes with other well known topological classifications of Cellular Automata such as Attractors, Languages and Equicontinuity classifications (Section 4). From the intersection classes we can provide necessary conditions for a cellular automaton to be universal (Section 5). Even in our model a universal cellular automaton is not minimal, not equicontinuous, does not have the shadowing property and, in particular, it is not regular. It is open also in our case the question whether a universal cellular automaton must have an infinite number of subsystems.

2. Notation and definitions

In this section, we introduce the notation and the basic concepts that will be necessary to understand the rest of the paper. Cellular Automata as dynamical systems were first considered by Hedlund in the late sixties who studied this formalism in the context of Symbolic Dynamics as endomorphisms of full shifts [10]. In this paper we will adopt Symbolic Dynamics terminology. For an introduction in Symbolic Dynamics the reader can refer to [19] and for an introduction on Topological Dynamics to [15]. In the following, we will assume that the reader is familiar with Computation theory and Formal Language theory (see, for example, [12]).

Let A be a finite alphabet with at least two elements. With $A^{\mathbb{Z}}$ and $A^{\mathbb{N}}$ we denote, respectively, the set of sequences $(x_i)_{i\in\mathbb{Z}}$ and $(x_i)_{i\in\mathbb{N}}$ where $x_i \in A$. For $x \in A^{\mathbb{Z}}$, let $x_{[i,j]} \in A^{j-i+1}$ denote the word $x_i x_{i+1} \ldots x_j$. We use the shortcut $w \sqsubset x$ to say that $w \in A^*$ is a subword of $x \in A^{\mathbb{Z}}$. Let us define a *metric* d on $A^{\mathbb{Z}}$ by

$$d(x,y) = 2^{-n} \quad \text{where} \quad n = min\{|i| \mid x_i \neq y_i\}|.$$

The set $A^{\mathbb{Z}}$ endowed with metric d is a compact metric space. For $u \in A^*$ and $i \in \mathbb{Z}$, let

$$[u]_i = \left\{x \in A^{\mathbb{Z}} \mid x_{[i,i+|u|-1]} = u\right\}$$

denote a *cylinder set*. Sometimes we will refer to the cylinder set $[u]_i$ simply with $[u]$. A cylinder set is a clopen (closed and open) set in $A^{\mathbb{Z}}$. Every clopen set in $A^{\mathbb{Z}}$ is a finite union of cylinder sets. The *shift maps* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ are defined by

$$\sigma(x)_i = x_{i+1}.$$

The shift map is a continuous function and it is bijective on $A^{\mathbb{Z}}$ while it is not on $A^{\mathbb{N}}$. The dynamical system $(A^{\mathbb{Z}},\sigma)$ is called *full shift*. A *shift space* or *subshift* is a non-empty closed subset $\Sigma \subseteq A^{\mathbb{Z}}$ which is also strongly shift invariant, i.e. $\sigma(\Sigma) = \Sigma$. A subshift is *one-sided* if it is a closed subset $\Sigma \subseteq A^{\mathbb{N}}$ and it is σ-invariant, i.e. $\sigma(\Sigma) \subseteq \Sigma$. Usually we will denote the *shift dynamical system* (Σ,σ) simply with Σ. A subshift Σ is *mixing* if for all clopen sets $U,V \subseteq \Sigma$, there exists $n_0 > 0$ such that for all $n \geqslant n_0$ $\sigma^n(U) \cap V \neq \emptyset$. The language associated to a subshift Σ is defined as

$$\mathcal{L}(\Sigma) = \{w \in A^* \mid \exists x \in \Sigma, w \sqsubset x\}.$$

Any subshift Σ is completely determined by the set of its *forbidden words* $A^* \setminus \mathcal{L}(\Sigma)$ (see [19]). A *shift of finite type* (SFT) is a subshift which can be defined by a *finite* set of forbidden words. The language $\mathcal{L}(\Sigma)$ of a subshift Σ is *bounded periodic*