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The alternation hierarchy in fixpoint logic with chop is strict too

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Abstract

Fixpoint logic with chop extends the modal μ-calculus with a sequential composition operator which results in an increase in expressive power. We develop a game-theoretic characterisation of its model checking problem and use these games to show that the alternation hierarchy in this logic is strict. The structure of this result follows the lines of Arnold's proof showing that the alternation hierarchy in the modal μ-calculus is strict over the class of binary trees.

Keywords: Modal logic; Expressive power; Games

1. Introduction

In 1996, Bradfield [3] and Lenzi [16] independently showed that the alternation hierarchy in the modal μ-calculus—multi-modal logic with extremal fixpoint quantifiers—is strict, i.e. there are certain formulas with nested alternating fixpoint quantifiers of depth n that are not equivalent to any formula with less than n alternating nested fixpoint quantifiers. Much earlier, Niwiński [18] already showed that there is a strict hierarchy w.r.t. expressiveness among formulas of the modal μ-calculus that do not contain the intersection operator.

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The importance of these results is motivated by the model checking problem for the modal μ-calculus. The best known algorithms are polynomial in the size of the structure and the size of the formula but still exponential in its alternation depth [7,20,9,19,6]. Furthermore, syntactic alternation makes formulas hard to read. Hence, a collapse of the alternation hierarchy could have led to simpler formulas that are easier to model check.

Equivalence of formulas and, thus, the issue of an expressive hierarchy, is only meaningful on a given class of structures. In the case of the modal μ-calculus, these are primarily transition systems. Note that the existence of such a hierarchy over a certain class of structures implies the existence over any superclass. Hence, one would like to have such a result over the "smallest" possible class of structures. In case of the modal μ-calculus these are binary trees. Note that because of invariance under bisimulation and the finite model property [11] it does not matter whether finite or infinite structures are considered.

Lenzi's proof originally works on n-ary trees for some fixed n. Since they can be encoded using binary trees, his result implies the existence of the alternation hierarchy over binary trees. Bradfield subsequently extended his proof to binary trees as well [4,5]. Note that the alternation hierarchy collapses over the class of linear structures [22,10,12].

Without a doubt the nicest proof establishing the hierarchy over binary trees was, however, given by Arnold [1]. First of all he shows the existence of a hierarchy w.r.t. expressiveness among parity tree automata, a special case of Rabin tree automata. Second, he uses the equivalence between the model checking problem for the modal μ-calculus and parity games: there are formulas of the modal μ-calculus which describe exactly those games that are won by either of the players. Finally, he uses Banach's fixpoint theorem on the metric space of binary trees to show that those formulas are hard for each level of the hierarchy, i.e. they are not equivalent to any formula on lower levels.

Those formulas are the so-called Walukiewicz formulas [23] that are simply a generalisation of the Emerson–Jutla [8] formulas and are very similar to the formulas that are shown to be hard in Bradfield's proof [4].

In 1999, Müller-Olm introduced *Fixpoint Logic with Chop* (FLC), which extends the modal μ-calculus with a sequential composition operator [17]. He showed that the expressive power of FLC reaches far beyond that of the modal μ-calculus. Despite this, its model checking problem remains decidable in deterministic, singly exponential time [15]. Again, the known model checking algorithms are exponential in the syntactic nesting depth of alternating fixpoint quantifiers [15,14]. Thus, it is fair to ask whether the alternation hierarchy within FLC is strict, too.

In the following we will answer this question to the affirmative. In order to do so, we adapt Arnold's proof for the strictness of the modal μ-calculus hierarchy over binary trees. In Section 2, we recall the syntax and semantics of FLC as well as its fragments of bounded alternation. Since there is no well-known correspondence to an automaton model we show the hierarchy result directly for the logic. This, however, requires a game-based characterisation of the model checking problem for FLC which we introduce and prove correct in Section 3. A preliminary version of these games with a misleading definition of winning condition has been published before [14]. Section 4 starts with another crucial ingredient to the hierarchy theorem: complementation closure. Given that the semantics of an FLC formula is a predicate transformer, it is not obvious that for every formula there is a complement. Yet the games of Section 3 provide a simple explanation that this is indeed the case. The hard part that follows proves the existence of formulas in FLC that describe exactly those FLC games that are won by one of the players. The rest of Section 4 finishes the hierarchy