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A semantic account of strong normalization in linear logic

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ABSTRACT

We prove that given two cut-free nets of linear logic, by means of their relational interpretations one can determine: 1) whether or not the net obtained by cutting the two nets is strongly normalizable, 2) (in case it is strongly normalizable) the maximum length of the reduction sequences starting from that net. As a by-product of our semantic approach, we obtain a new proof of the conservation theorem for Multiplicative Exponential Linear Logic (*MELL*) which does not rely on confluence; this yields an alternative proof of strong normalization for *MELL*.

1. Introduction

Linear Logic (LL, [19]) originated from the coherent model of typed λ-calculus: the category of coherent spaces and linear maps was "hidden" behind the category of coherent spaces and stable maps. It then turned out that the coherence relation was not necessary to interpret linear logic proofs (proof-nets), and this remark led to the so-called multiset based relational model of LL: the interpretation of proof-nets in the category **Rel** of sets and relations. Since then, many efforts have been done to understand to which extent the relational interpretation of a proof-net is nothing but a different representation of the proof itself: in Girard's original paper ([19]), with every proof-net was associated the set of "results of experiments" of the proof-net, a set proven to be invariant with respect to cut elimination. Later on these "results" have been represented as nets themselves, and through Taylor's expansion a proof-net can been represented as an infinite linear combination of nets (see [17] and [18]). On the other hand, the first author proved in [12] that one can always recover, from the relational interpretation of a cut-free proof-net, the proof-net itself.

This paper establishes another tight link between the relational model and LL proof-nets. We follow the approach to the semantics of bounded time complexity consisting in measuring by semantic means the execution of any program, regardless of its computational complexity. The aim is to compare different computational behaviours and to learn something afterwards on the very nature of bounded time complexity. Following this approach and inspired by [16], in [10,11] one of the authors of the present paper could determine the execution time of an untyped λ-term from its interpretation in the Kleisli category of the comonad associated with the finite multisets functor on the category of sets and relations. Such an interpretation is the same as the interpretation of the net encoding the λ-term in the multiset based relational model of linear logic. The execution time is measured there in terms of elementary steps of the so-called Krivine machine. Also, [10,11] give a precise relation between an intersection types system introduced in [7] and experiments in the multiset based relational model. Experiments are a tool introduced by Girard in [19] allowing to enumerate the interpretation of proofs pointwise. An experiment corresponds to a type derivation and the result of an experiment corresponds to a type.

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This same approach was applied in [13] to LL to show how it is possible to obtain the number of steps of cut elimination by semantic means (notice that the measure being now the number of cut elimination steps, here is a first difference with [10,11] where Krivine's machine was used to measure execution time). The results of [13] are presented in the framework of proof-nets, that we call nets in this paper: if π' is a net obtained by applying some steps of cut elimination to π, the main property of any model is that the interpretation $[\![\pi]\!]$ of π is the same as the interpretation $[\![\pi']\!]$ of π', so that from $[\![\pi]\!]$ it is clearly impossible to determine the number of steps leading from π to π'. Nevertheless, in [13] it is shown that if π_1 and π_2 are two cut-free nets connected by means of a cut-link, one can answer the two following questions by only referring to the interpretations $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ in the relational model:

- is it the case that the net obtained by cutting π_1 and π_2 is weakly normalizable?
- if the answer to the previous question is positive, what is the number of cut reduction steps leading from the net with cut to a cut-free one?

In the present paper, still by only referring to the interpretations $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ in the relational model, we answer the two following variants of the previous questions:

1. is it the case that the net obtained by cutting π_1 and π_2 is strongly normalizable?
2. if the answer to the previous question is positive, what is the maximum length (i.e. the number of cut reduction steps) of the reduction sequences starting from the net obtained by cutting π_1 and π_2?

Despite the fact that the new questions are just little variations on the old ones, the answers *are not* variants of the old ones, and require the development of new tools (see for example the new $(\!|\,|\!)$-interpretation of Definition 20). The first question makes sense only in an untyped framework (in the typed case, cut elimination is strongly normalizing, see [19,8,22] and... Subsection 4.3!), and we thus study in Section 2 nets and their stratified reduction in an untyped framework. Subsection 2.1 mainly recalls definitions and notations coming from [13], while in Subsection 2.2, we prove two syntactic results that will be used in the sequel: 1) Proposition 10 reduces strong normalization to "non-erasing" strong normalization (and will be used in Section 4), and 2) Proposition 16 shows that when a net is strongly normalizable there exists a "canonical" reduction sequence of maximum length, consisting first of "stratified non-erasing" steps and then of "erasing antistratified" steps (and will be used in Section 5).

In Section 3, we introduce the standard notion of experiment (called $[\![\,]\!]$-experiment in this paper) leading to the usual interpretation (called $[\![\,]\!]$-interpretation in this paper) of a net in the category of sets and relations (the multiset based relational model of linear logic). In the same Definition 20, we introduce $(\!|\,|\!)$-experiments, leading to the $(\!|\,|\!)$-interpretation of nets: the main difference between $[\![\,]\!]$-experiments and $(\!|\,|\!)$-experiments is the behaviour w.r.t. weakening links. And indeed, the main difference between weak and strong normalization lies in the fact that to study the latter property we cannot "forget pieces of proofs" (and this is actually what the usual $[\![\,]\!]$-interpretation does by assigning the empty multiset as label to the conclusion of weakening links). The newly defined $(\!|\,|\!)$-interpretation *does not* yield a model of linear logic: it is invariant only w.r.t. *non-erasing* reduction steps (Proposition 24).

In Section 4, we point out an intrinsic difference between the semantic characterization of strong normalization and the one of weak normalization proven in [13] (here Theorem 36): there exist nets π and π' such that $[\![\pi]\!] = [\![\pi']\!]$ and π is strongly normalizing while π' is not, which clearly shows that there is no hope (in the general case) to extract the information on the strong normalizability of a net from its $[\![\,]\!]$-interpretation (Remark 4). We then prove that in case π is a cut-free net, its $(\!|\,|\!)$-interpretation $(\!|\pi|\!)$ can be computed from its "good old" $[\![\,]\!]$-interpretation $[\![\pi]\!]$ (Proposition 31). This implies that to answer Questions 1 and 2 by only referring to the interpretations $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ in the "good old" relational model of linear logic, we are allowed to use the newly defined $(\!|\,|\!)$-interpretations $(\!|\pi_1|\!)$ and $(\!|\pi_2|\!)$. We then accurately adapt the notion of size of an $[\![\,]\!]$-experiment of the relational model to $(\!|\,|\!)$-experiments, in order to obtain a variant of the "Key Lemma" (actually Lemmata 17 and 20) of [13]: Lemma 35 measures the difference between the size of (suitable) experiments of a net and the size of (suitable) experiments of any of its one step reducts. We can thus answer Question 1 (Corollary 40).

Our qualitative results of Subsection 4.2 allow to give a new proof of the so called "Conservation Theorem" (here Theorem 42) for Multiplicative Exponential Linear Logic (*MELL*). Such a result is a crucial step in the traditional proof of strong normalization for Linear Logic ([19,8,22]) and it is usually proven using confluence ([8,22]): our semantic approach does not rely on confluence and yields thus a proof of strong normalization for *MELL* which does not use confluence (Corollary 47 of Subsection 4.3, see also Remark 13).

In Section 5, we answer Question 2: thanks to Proposition 16 it is enough from $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ to predict the length of a "canonical" reduction sequence, and by Proposition 31 we can substitute $(\!|\pi_1|\!)$ and $(\!|\pi_2|\!)$ for $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$. We first measure the length of the longest "non-erasing stratified" reduction sequence, by means of the size of (suitable) experiments, and we then shift to the size of results of $(\!|\,|\!)$-experiments, that is elements of the $(\!|\,|\!)$-interpretation. We then measure the length of the longest "erasing antistratified" reduction sequence starting from a "non-erasing normal" net, relating this length to the number of (erasing) cuts of the net, and counting this number using the $(\!|\,|\!)$-interpretation. The precise answer to Question 2 is Theorem 57. We end the section by giving a concrete example (Example 58), showing also that only a little