Information and Computation 240 (2015) 42–55

Contents lists available at ScienceDirect

Information and Computation

www.elsevier.com/locate/yinco

Boolean matrices with prescribed row/column sums and stable homogeneous polynomials: Combinatorial and algorithmic applications

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ARTICLE INFO

Article history:
Received 6 October 2013
Available online 16 October 2014

Keywords:
Permanent
Algorithm
Convex
Stable polynomial
Complexity

ABSTRACT

We prove a new efficiently computable lower bound on the coefficients of stable homogeneous polynomials and present its algorithmic and combinatorial applications. Our main application is the first poly-time deterministic algorithm which approximates the partition functions associated with boolean matrices with prescribed row and column sums within simply exponential multiplicative factor. This new algorithm is a particular instance of new polynomial time deterministic algorithms related to the multiple partial differentiation of polynomials given by evaluation oracles.

1. Basic definitions and motivations

For given two integer vectors $\mathbf{r} = (r_1, ..., r_n)$ and $\mathbf{c} = (c_1, ..., c_m)$, we denote as $BM_{\mathbf{r},\mathbf{c}}$ the set of boolean $n \times m$ matrices with prescribed row sums $\mathbf{r}$ and column sums $\mathbf{c}$.

Next, we introduce an analogue of the permanent (a partition function associated with $BM_{\mathbf{r},\mathbf{c}}$):

$$PE_{\mathbf{r},\mathbf{c}}(A) =: \sum_{B \in BM(\mathbf{r},\mathbf{c})} \prod_{1 \le i \le n; 1 \le j \le m} A(i,j)^{B(i,j)}, \tag{1}$$

where A is an $n \times m$ complex matrix. We suppose that $0^0 = 1$. Note that if A is an $n \times n$ matrix; $\mathbf{r} = \mathbf{c} = e_n$, where e_n is an n-dimensional vector of all ones, then the definition (1) reduces to the permanent: $PE_{e_n,e_n}(A) = \mathrm{per}(A)$.

The main focus of this note is on bounds and **deterministic** algorithms for $PE_{\mathbf{r},\mathbf{c}}(A)$ in the case that the matrix A is real and non-negative. To avoid messy formulas, we will mainly focus below on the uniform square case, i.e. $n = m$ and $r_i = c_j = r$, $1 \le i, j \le n$ and use simplified notations:

$$BM_{re_n,re_n} =: BM(r,n); \qquad PE_{re_n,re_n}(A) =: PE(r,A),$$

where e_n denotes the vector $(1, \dots, 1)$. Boolean matrices with prescribed row and column sums are one of the most classical and intensely studied topics in analytic combinatorics, with applications to many areas from applied statistics to the representation theory. We, and many other researchers, are interested in the counting aspect, i.e. in computing/bounding/approximating the partition function $PE_{\mathbf{r},\mathbf{c}}(A)$. It was known already to W.T. Tutte [18] that this partition function can be in poly-time reduced to the permanent. Therefore, if A is nonnegative the famous **FPRAS** [14] can be applied, as was

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http://dx.doi.org/10.1016/j.ic.2014.09.007
0890-5401/

already mentioned in [14] as one of the main applications. We are after deterministic poly-time algorithms. A. Barvinok initiated this, deterministic, line of algorithmic research in [1]. He also used the reduction to the permanent and the van der Waerden–Falikman–Egorychev (**VFE**) [5,3] celebrated lower bound on the permanent of doubly-stochastic matrices:

$$\mathrm{per}(A) \geq \mathrm{vdw}(n) =: \frac{n!}{n^n}.$$

The techniques in [1] result in a deterministic poly-time algorithm approximating $PE(r, A)$ within multiplicative factor $(\Omega(\sqrt{n}))^n$ for any fixed r, even for $r = 1$. Such poor approximation is due the fact that the reduction to the permanent produces highly structured $n^2 \times n^2$ matrices. **VFE** bound is clearly a powerful algorithmic tool, as was recently effectively illustrated in [19]. Yet, neither **VFE** nor even more refined Schrijver's lower bound [17] are sharp enough for those structured matrices. This phenomenon was observed by A. Schrijver 30 years ago in [16]. The author introduced in [9] and [10] a new approach to lower bounds. We will give a brief description of the approach and refine it. The new lower bounds are asymptotically sharp and allow, for instance, to get a deterministic poly-time algorithm to approximate $PE(r, A)$ within multiplicative factor $f(r)^{-1}$ where

$$f(r) = \left(\frac{\mathrm{vdw}(n)}{\mathrm{vdw}(r)\,\mathrm{vdw}(n-r)}\right)^{n\frac{r-1}{r}} \frac{\mathrm{vdw}(n)}{\mathrm{vdw}(r)^{\frac{n}{r}}} \approx \left(\sqrt{2\pi \min(r, n-r)}\right)^{-n}.$$

We use the symbol $\approx$ *to ignore all sub-exponential terms.*

Besides, we show the algorithm from [1] actually approximates within (roughly) multiplicative factor $f(r)^{-2}$. So, for fixed r or $n - r$ the new bounds give simply exponential factor. But, say for $r = \frac{n}{2}$, the current factor is not simply exponential. *Is there a deterministic Non-Approximability result for* $PE(\frac{n}{2}, A)$*?*

We also study the sparse case, i.e. when, say, the columns of matrix A have relatively few non-zero entries. In this direction we generalize, reprove, sharpen the results of A. Schrijver [16] on how many k-regular subgraphs $2k$-regular bipartite graph can have.

The main message of this paper is that when one needs to deal with the permanent of highly structured matrices the only (and often painless) way to get sharp lower bounds is to use **stable polynomials approach**. Prior to [9] and [10] **VFE** this was, essentially, the only general purpose non-trivial lower bound on the permanent. It is not the case anymore.

1.1. Generating polynomials

The goal of this subsection is to represent $PE_{\mathbf{r},\mathbf{c}}(A)$ as a coefficient of some effectively computable polynomial.

1. The following natural representation in the case of unit weights, i.e. $A(i, j) \equiv 1$, was already in [15], the general case of it was used in [1].

$$PE_{\mathbf{r},\mathbf{c}}(A) = \left[\prod_{1\leq i\leq n} y_i^{r_i} \prod_{1\leq j\leq m} x_j^{c_j}\right] \prod_{1\leq i\leq n, 1\leq j\leq m} \left(1 + A(i, j)x_j y_i\right), \tag{2}$$

i.e. $PE_{\mathbf{r},\mathbf{c}}(A)$ is the coefficient of the monomial $\prod_{1\leq i\leq n} y_i^{r_i} \prod_{1\leq j\leq m} x_j^{c_j}$ in the non-homogeneous polynomial $\prod_{1\leq i\leq n, 1\leq j\leq m}(1 + A(i, j)x_j y_i)$.

It is easy to convert the non-homogeneous formula (2) into a homogeneous one:

$$PE_{\mathbf{r},\mathbf{c}}(A) = \left[\prod_{1\leq j\leq m} x_j^{c_j} \prod_{1\leq i\leq n} z_i^{m-r_i}\right] \prod_{1\leq i\leq n, 1\leq j\leq m} \left(z_i + A(i, j)x_j\right). \tag{3}$$

As the polynomial $\prod_{1\leq i\leq n, 1\leq j\leq m}(z_i + A(i, j)x_j)$ is a product of linear forms, the formula (3) allows us to express $PE_{\mathbf{r},\mathbf{c}}(A)$ as the permanent of some $nm \times nm$ matrix, the fact essentially proved in a very different way in [18]. The permanent also showed up, in a similar context of Eulerian Orientations, in [16].

Indeed, associate with any $k \times l$ matrix B the product polynomial

$$Prod_B(x_1, \ldots, x_l) =: \prod_{1\leq i\leq k} \sum_{1\leq j\leq l} B(i, j)x_j. \tag{4}$$

Then

$$\left[\prod_{1\leq j\leq l} x_j^{\omega_j}\right] Prod_B(x_1, \ldots, x_l) = \mathrm{per}(B_{\omega_1,\ldots,\omega_l}) \prod_{1\leq j\leq l} (\omega_j!)^{-1}, \tag{5}$$

where the $k \times k$ matrix $B_{\omega_1,\ldots,\omega_l}$ consists of ω_j copies of the jth column of B, $1 \leq j \leq l$. We remind the reader that well-known formula (4) easily follows from the following obvious identity: $Prod_B(x_1, \ldots, x_l)$ equals

$$(k!)^{-1}\mathrm{per}\left(\left[x_1 B^{(1)} + \cdots + x_l B^{(l)}, \ldots, x_1 B^{(1)} + \cdots + x_l B^{(l)}\right]\right),$$

where $B^{(j)}$ is the jth column of B.