Information and Computation 225 (2013) 1–15

Contents lists available at SciVerse ScienceDirect

Information and Computation

www.elsevier.com/locate/yinco

Subshifts as models for MSO logic ☆

Emmanuel Jeandel [a], Guillaume Theyssier [b,*]

[a] LORIA, UMR 7503, Campus Scientifique, BP 239, 54 506 Vandoeuvre-lès-Nancy, France
[b] LAMA (Université de Savoie, CNRS), Campus Scientifique, 73376 Le Bourget-du-lac cedex, France

ARTICLE INFO

Article history:
Received 7 December 2009
Revised 24 June 2011
Available online 30 January 2013

Keywords:
Symbolic dynamics
Model theory
Tilings

ABSTRACT

We study the Monadic Second Order (MSO) Hierarchy over colorings of the discrete plane, and draw links between classes of formula and classes of subshifts. We give a characterization of existential MSO in terms of projections of tilings, and of universal sentences in terms of combinations of "pattern counting" subshifts. Conversely, we characterize logic fragments corresponding to various classes of subshifts (subshifts of finite type, sofic subshifts, all subshifts). Finally, we show by a separation result how the situation here is different from the case of tiling pictures studied earlier by Giammarresi et al.

1. Introduction

There is a close connection between words and monadic second-order (MSO) logic. Büchi and Elgot proved for finite words that MSO-formulas correspond exactly to regular languages. This relationship was developed for other classes of labeled graphs; trees or infinite words enjoy a similar connection. See [1,2] for a survey of existing results. Colorings of the entire plane, i.e. tilings, represent a natural generalization of biinfinite words to higher dimensions, and as such enjoy similar properties. We plan to study in this paper tilings for the point of view of monadic second-order logic.

From a computer science point of view, tilings and more generally subshifts are the underlying objects of several computing models including cellular automata [3–5], Wang tiles [6,7] and self-assembly tilings [8,9]. Following the recent trend to better understand such 'natural computing models', one of the motivations of the present paper is to extend towards these models the fruitful links established between languages of finite words and MSO logic.

Tilings and logic have a shared history. The introduction of tilings can be traced back to Hao Wang [10], who introduced his celebrated tiles to study the (un)decidability of the $\forall\exists\forall$ fragment of first-order logic. The undecidability of the domino problem by his PhD student Berger [11] lead then to the undecidability of this fragment [12]. Seese [13,14] used the domino problem to prove that graphs with a decidable MSO theory have a bounded tree width. Makowsky [15,16] used the construction by Robinson [17] to give the first example of a finitely axiomatizable theory that is super-stable. More recently, Oger [18] gave generalizations of classical results on tilings to locally finite relational structures. See the survey [19] for more details.

Previously, a finite variant of tilings, called tiling pictures, was studied [20,21]. Tiling pictures correspond to colorings of a *finite* region of the plane, this region being bordered by special '#' symbols. It is proven for this particular model that language recognized by EMSO-formulas corresponds exactly to so-called finite tiling systems, i.e. projections of finite tilings.

☆ The authors are partially supported by ANR-09-BLAN-0164.
* Corresponding author.
E-mail addresses: emmanuel.jeandel@loria.fr (E. Jeandel), guillaume.theyssier@univ-savoie.fr (G. Theyssier).

0890-5401/$ – see front matter
http://dx.doi.org/10.1016/j.ic.2013.01.003

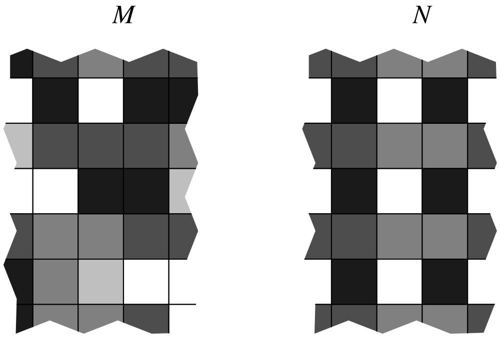

Fig. 1. Two configurations.

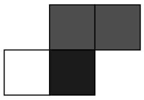

Fig. 2. A pattern P. P appears in M but presumably not in N.

The equivalent of finite tiling systems for infinite pictures are so-called *sofic subshifts* [22]. A *sofic subshift* represents intuitively local properties and ensures that every point of the plane behaves in the same way. As a consequence, there is no general way to enforce that some specific color, say **A**, appears at least once. Hence, some simple first-order existential formulas have no equivalent as sofic subshift (and even subshift). This is where the border of # for finite pictures plays an important role: Without such a border, results on finite pictures would also stumble on this issue. See [23] for similar results on finite pictures without borders.

We deal primarily in this article with subshifts. See [24] for other acceptance conditions (what we called subshifts of finite type correspond to A-acceptance in this paper).

Finally, note that all decision problems in our context are non-trivial: To decide if a universal first-order formula is satisfiable (the domino problem, presented earlier) is not recursive. Worse, it is Σ_1^1-hard to decide if a tiling of the plane exists where some given color appears infinitely often [25,24]. As a consequence, the satisfiability of MSO-formulas is at least Σ_1^1-hard.

In this paper, we will prove how various classes of formula correspond to well-known classes of subshifts. Some of the results of this paper were already presented in [26].

2. Symbolic spaces and logic

2.1. Configurations

Consider the discrete lattice $\mathbb{Z}^2$. For any finite set Q, a Q-configuration is a function from $\mathbb{Z}^2$ to Q. Q may be seen as a set of *colors* or *states*. An element of $\mathbb{Z}^2$ will be called a *cell*. A configuration will usually be denoted C, M or N.

Fig. 1 shows an example of two different configurations of $\mathbb{Z}^2$ over a set Q of 5 colors. As a configuration is infinite, only a finite fragment of the configurations is represented in the figure. We choose not to represent which cell of the picture is the origin $(0,0)$. This will indeed be of no importance as we use only translation invariant properties.

For any $z \in \mathbb{Z}^2$ we denote by σ_z the *shift* map of vector z, i.e. the function from Q-configurations to Q-configurations such that for all $C \in Q^{\mathbb{Z}^2}$:

$$\forall z' \in \mathbb{Z}^2, \sigma_z(C)\big(z'\big) = C\big(z' - z\big)$$

A *pattern* is a partial configuration. A pattern $P : X \to Q$ where $X \subseteq \mathbb{Z}^2$ occurs in $C \in Q^{\mathbb{Z}^2}$ at position z_0 if

$$\forall z \in X, C(z_0 + z) = P(z)$$

We say that P occurs in C if it occurs at some position in C. As an example the pattern P of Fig. 2 occurs in the configuration M but not in N (or more accurately not on the finite fragment of N depicted in the figure). A finite pattern is a partial configuration of finite domain. All patterns in the following will be finite. The *language* $\mathcal{L}(C)$ of a configuration C is the set of finite patterns that occur in C. We naturally extend this notion to sets of configurations.

A *subshift* is a natural concept that captures both the notion of *uniformity* and *locality*: the only description "available" from a configuration C is the finite patterns it contains, that is $\mathcal{L}(C)$. Given a set $\mathcal{F}$ of patterns, let $X_{\mathcal{F}}$ be the set of all configurations where no patterns of $\mathcal{F}$ occur.