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Distributed coloring algorithms for triangle-free graphs ☆

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ABSTRACT

Vertex coloring is a central concept in graph theory and an important symmetry-breaking primitive in distributed computing. Whereas degree-Δ graphs may require palettes of $\Delta+1$ colors in the worst case, it is well known that the chromatic number of many natural graph classes can be much smaller. In this paper we give new distributed algorithms to find (Δ/k)-coloring in graphs of girth 4 (triangle-free graphs), girth 5, and trees. The parameter k can be at most $(\frac{1}{4}-o(1))\ln\Delta$ in triangle-free graphs and at most $(1-o(1))\ln\Delta$ in girth-5 graphs and trees, where $o(1)$ is a function of Δ. Specifically, for Δ sufficiently large we can find such a coloring in $O(k+\log^* n)$ time. Moreover, for *any* Δ we can compute such colorings in roughly logarithmic time for triangle-free and girth-5 graphs, and in $O(\log\Delta+\log_\Delta \log n)$ time on trees. As a byproduct, our algorithm shows that the chromatic number of triangle-free graphs is at most $(4+o(1))\frac{\Delta}{\ln\Delta}$, which improves on Jamall's recent bound of $(67+o(1))\frac{\Delta}{\ln\Delta}$. Finally, we show that $(\Delta+1)$-coloring for triangle-free graphs can be obtained in sublogarithmic time for any Δ.

1. Introduction

A proper t-coloring of a graph $G=(V,E)$ is an assignment from V to $\{1,\dots,t\}$ (colors) such that no edge is monochromatic, or equivalently, each color class is an independent set. The *chromatic number* $\chi(G)$ is the minimum number of colors needed to properly color G. Let Δ be the maximum degree of the graph. It is easy to see that sometimes $\Delta+1$ colors are necessary, e.g., on an odd cycle or a $(\Delta+1)$-clique. Brooks' celebrated theorem [9] states that these are the *only* such examples and that every other graph can be Δ-colored. Vizing [38] asked whether Brooks' Theorem can be improved for triangle-free graphs. In the 1970s Borodin and Kostochka [8], Catlin [10], and Lawrence [27] independently proved that $\chi(G)\le\frac{3}{4}(\Delta+2)$ for triangle-free G, and Kostochka (see [20]) improved this bound to $\chi(G)\le\frac{2}{3}(\Delta+2)$.

Existential bounds Better asymptotic bounds were achieved in the 1990s by using an iterated approach, often called the "Rödl Nibble". The idea is to color a very small fraction of the graph in a sequence of rounds, where after each round some property is guaranteed to hold with some small non-zero probability. Kim [22] proved that in any girth-5 graph G, $\chi(G)\le(1+o(1))\frac{\Delta}{\ln\Delta}$. This bound is optimal to within a factor-2 under *any* lower bound on girth. (Constructions of Kostochka and Mazurova [24] and Bollobás [7] show that there is a graph G of arbitrarily large girth and $\chi(G)>\frac{\Delta}{2\ln\Delta}$.) Building

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on [22], Johansson (see [30]) proved that $\chi(G)=O(\frac{\Delta}{\ln\Delta})$ for any triangle-free (girth-4) graph G.[1] In relatively recent work Jamall [17] proved that the chromatic number of triangle-free graphs is at most $(67+o(1))\frac{\Delta}{\ln\Delta}$.

Algorithms We assume the $\mathcal{LOCAL}$ model [33] of distributed computation. In this model, vertices host processors which operate in synchronized rounds; vertices can communicate one arbitrarily large message across each edge in each round; local computation is free; *time* is measured by the number of rounds. Grable and Panconesi [15] gave a distributed algorithm that Δ/k-colors a girth-5 graph in $O(\log n)$ time, where $\Delta > \log^{1+\epsilon} n$ and $k \le \delta \ln\Delta$ for any $\epsilon > 0$ and some $\delta < 1$ depending on ϵ.[2] Jamall [18] showed a sequential algorithm for $O(\Delta/\ln\Delta)$-coloring a triangle-free graph in $O(n\Delta^2\ln\Delta)$ time, for any $\epsilon > 0$ and $\Delta > \log^{1+\epsilon} n$.

Note that there are *two* gaps between the existential [22,30,17] and algorithmic results [15,18]. The algorithmic results use a constant factor more colors than necessary (compared to the existential bounds) and they only work when $\Delta \ge \log^{1+\Omega(1)} n$ is sufficiently large, whereas the existential bounds hold for all Δ.

New results We give new distributed algorithms for (Δ/k)-coloring triangle-free graphs that simultaneously improve on both the existential and algorithmic results of [15,30,17,18]. Our algorithms run in $\log^{1+o(1)} n$ time for *all* Δ and in $O(k+\log^* n)$ time for Δ sufficiently large. Moreover, we prove that the chromatic number of triangle-free graphs is $(4+o(1))\frac{\Delta}{\ln\Delta}$.

Theorem 1. *Fix a constant $\epsilon > 0$. Let Δ be the maximum degree of a triangle-free graph G, assumed to be at least some Δ_ϵ depending on ϵ. Let $k \ge 1$ be a parameter such that $k \le \frac{1}{4}(1-2\epsilon)\ln\Delta$. Then G can be (Δ/k)-colored, in time $O(k+\log^*\Delta)$ if $\Delta^{1-\frac{4k}{\ln\Delta}-\epsilon} = \Omega(\ln n)$, and, for any Δ, in time on the order of*

$$\left(k+\log^*\Delta\right)\cdot\frac{\ln n}{\Delta^{1-\frac{4k}{\ln\Delta}-\epsilon}}\cdot\exp\!\left(O(\sqrt{\ln\ln n})\right) = (\ln n)^{1+O(1/\sqrt{\ln\ln n})} = \ln^{1+o(1)} n.$$

The first time bound comes from an $O(k+\log^*\Delta)$-round procedure, each round of which succeeds with probability $1-1/\mathrm{poly}(n)$. However, as Δ decreases the probability of failure tends to 1. To enforce that each step succeeds with high probability we use a version of the Local Lemma algorithm of Moser and Tardos [31] optimized for the parameters of our problem.

Theorem 1 has a complex tradeoff between the minimum threshold Δ_ϵ, the number of colors, and the threshold for Δ beyond which the running time becomes $O(\log^* n)$. The following corollaries highlight some interesting parameterizations of Theorem 1.

Collorary 1. *The chromatic number of triangle-free graphs with maximum degree Δ is at most $(4+o(1))\Delta/\ln\Delta$.*

Proof. Fix an $\epsilon' > 0$ and choose $k = \ln\Delta/(4+\epsilon')$ and $\epsilon = \epsilon'/(2(4+\epsilon'))$. Theorem 1 states that for Δ at least some $\Delta_{\epsilon'}$, the chromatic number is at most $(4+\epsilon')\Delta/\ln\Delta$. Now let $\epsilon' = o(1)$ be a function of Δ tending slowly to zero. (The running time of the algorithm that finds such a coloring is never more than $(\ln n)^{1+O(1/\sqrt{\ln\ln n})}$.) □

Collorary 2. *Fix any $\delta > 0$. A $(4+\delta)\Delta/\ln\Delta$-coloring of an n-vertex triangle-free graph can be computed in $O(\log^* n)$ time, provided $\Delta > (\ln n)^{(4+\delta)\delta^{-1}+o(1)}$ and n is sufficiently large.*

Proof. Set $k = \ln\Delta/(4+\delta)$ and let $\epsilon = o(1)$ tend slowly to zero as a function of n. If we have

$$\Delta^{1-4k/\ln\Delta-\epsilon} = \Delta^{1-4/(4+\delta)-\epsilon} = \Delta^{\delta(4+\delta)^{-1}-\epsilon} = \Omega(\ln n),$$

or equivalently, $\Delta > (\ln n)^{\delta^{-1}(4+\delta)+o(1)}$, then a $(4+\delta)\Delta/\ln\Delta$-coloring can be computed in $O(\log^* n)$ time. (For n sufficiently large and ϵ tending slowly enough to zero, the lower bound on Δ also implies $\Delta > \Delta_\epsilon$.) □

Theorem 1 also shows that some colorings can be computed in *sub*logarithmic time, even when Δ is too small to achieve an $O(\log^* n)$ running time.

Collorary 3. *Fix a $\delta > 0$ and let $k = o(\ln\Delta)$. If $\Delta > (\ln n)^\delta$, a (Δ/k)-coloring can be computed in $(\ln n)^{1-\delta+o(1)}$ time.*

[1] We are not aware of any extant copy of Johansson's manuscript. It is often cited as a DIMACS Technical Report, though no such report exists. Molloy and Reed [30] reproduced a variant of Johansson's proof showing that $\chi(G) \le 160\frac{\Delta}{\ln\Delta}$ for triangle-free G.

[2] They claimed that their algorithm could also be extended to triangle-free graphs. Jamall [18] pointed out a flaw in their argument.