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A Kraft–McMillan inequality for free semigroups of upper-triangular matrices

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ABSTRACT

We prove a version of the Kraft–McMillan inequality for free semigroups of upper-triangular integer matrices of any dimension.

1. Introduction

Let S be a semigroup and let X be a subset of S. Then X is called a *code* if for any integers $m, n \geq 1$ and any elements $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$ the equation

$$x_1 x_2 \ldots x_m = y_1 y_2 \ldots y_n$$

implies that

$$m = n \quad \text{and} \quad x_i = y_i \quad \text{for } 1 \leq i \leq m.$$

A semigroup S is called *free* if there exists a code $X \subseteq S$ such that X generates S. For a general introduction to questions concerning freeness of various semigroups see [3].

Let now Σ be an alphabet and let Σ^* be the free monoid generated by Σ. The free submonoids of Σ^* and the codes generating them have been extensively studied. For an excellent introduction to this theory, sometimes called the theory of variable-length codes, see [1].

Most results concerning codes in word monoids do not extend to more general monoids. As an example, consider the monoid $\mathrm{Tri}(3, \mathbb{N})$ of upper-triangular 3×3 matrices having nonnegative integer entries. While in a word monoid a two-element set is a code if and only if the two words are not powers of a common word, the matrices

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are not powers of a common matrix but the set $\{A, B\}$ still fails to be a code because we have $AB = BA$.

Also, in a word monoid it is easy to decide whether or not a given finite set is a code, while in $\mathrm{Tri}(3, \mathbb{N})$ this problem is undecidable (see [1,2]).

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Let again Σ be an alphabet and assume that $\pi : \Sigma^* \to \mathbb{R}_+$ is a Bernoulli distribution. (Here $\mathbb{R}_+$ is the multiplicative monoid of nonnegative real numbers.) In other words, π is a monoid morphism such that

$$\sum_{\sigma \in \Sigma} \pi(\sigma) = 1.$$

If $C \subseteq \Sigma^*$, define

$$\pi(C) = \sum_{c \in C} \pi(c).$$

Now, if C is a code, then the Kraft–McMillan inequality says that

$$\pi(C) \le 1$$

(see [1,4,5]). This result can be applied to show that a given subset C of Σ^* is not a code and, more importantly, it implies that if a code C satisfies $\pi(C) = 1$ then C is a maximal code. Recall that a code C is *maximal* if there does not exist a code D such that C is a proper subset of D.

The purpose of this paper is to prove a version of the Kraft–McMillan inequality for codes in the monoid of upper-triangular integer matrices of any dimension. We will not use Bernoulli distributions. Instead, as a measure of a matrix we will use its spectral radius. The proof of the inequality will use heavily special properties of upper-triangular matrices. It turns out that the applicability of our inequality is greatly enhanced by the fact that the spectral radius $\rho(MN)$ of the product of two upper-triangular matrices M and N is often less than the product of $\rho(M)$ and $\rho(N)$.

We now outline the contents of the paper. In Section 2 we recall the basic definitions. In Section 3 we state and prove the Kraft–McMillan inequality for codes consisting of upper-triangular integer matrices. In Section 4 we present various examples and explain the connection of our inequality to the classical Kraft–McMillan inequality.

We will assume some familiarity with elementary linear algebra, but otherwise the paper is self-contained.

2. Definitions

As usual, $\mathbb{Z}$ is the set of integers.

If k is a positive integer, $\mathbb{Z}^{k\times k}$ is the set of $k \times k$ matrices having integer entries and $\mathrm{Tri}(k, \mathbb{Z})$ is the subset of $\mathbb{Z}^{k\times k}$ consisting of upper-triangular matrices.

If M is a $k \times k$ matrix, then the (i, j)-entry of M is denoted by M_{ij} for $1 \le i, j \le k$ and the *diagonal* of M, denoted by $\mathrm{diag}(M)$, is the vector $(M_{11}, M_{22}, \dots, M_{nn})$.

If $M \in \mathbb{Z}^{k\times k}$ and the characteristic values of M are $\lambda_1, \dots, \lambda_k$, then the *spectral radius* of M is defined by

$$\rho(M) = \max\{|\lambda_1|, \dots, |\lambda_k|\},$$

where $|\lambda|$ is the absolute value of λ. If $M \in \mathrm{Tri}(k, \mathbb{Z})$, then the characteristic values of M are the diagonal entries M_{ii} for $1 \le i \le k$ and

$$\rho(M) = \max\{|M_{11}|, |M_{22}|, \dots, |M_{kk}|\}.$$

A matrix $M \in \mathbb{Z}^{k\times k}$ is called *nilpotent* if there is a positive integer n such that M^n is the zero matrix. A matrix $M \in \mathrm{Tri}(k, \mathbb{Z})$ is nilpotent if and only if its diagonal is the zero vector, or equivalently, if and only if $\rho(M) = 0$.

Suppose $\mathcal{M} \subseteq \mathbb{Z}^{k\times k}$. If n is a nonnegative integer then $\mathcal{M}^n$ is the set defined by

$$\mathcal{M}^n = \{M_1 M_2 \dots M_n \mid M_1, \dots, M_n \in \mathcal{M}\}$$

and $\mathcal{M}^*$ is the set

$$\mathcal{M}^* = \bigcup_{n\ge 0} \mathcal{M}^n.$$

A set $\mathcal{M} \subseteq \mathbb{Z}^{k\times k}$ is called a *code* if for all $s, t \ge 1$ and $M_1, \dots, M_s, N_1, \dots, N_t \in \mathcal{M}$ the condition

$$M_1 M_2 \dots M_s = N_1 N_2 \dots N_t$$

implies that

$$s = t \quad \text{and} \quad M_i = N_i \quad \text{for all } 1 \le i \le s.$$

If k is a positive integer, a code $\mathcal{M} \subseteq \mathrm{Tri}(k, \mathbb{Z})$ is called *maximal* if there does not exist a code $\mathcal{N} \subseteq \mathrm{Tri}(k, \mathbb{Z})$ such that $\mathcal{M}$ is a proper subset of $\mathcal{N}$. (Observe that we consider maximality in the monoid $\mathrm{Tri}(k, \mathbb{Z})$.)