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A new mapping between combinatorial proofs and sequent calculus proofs read out from logical flow graphs

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ABSTRACT

Combinatorial proofs are abstract invariants for sequent calculus proofs, similarly to homotopy groups which are abstract invariants for topological spaces. Sequent calculus fails to be surjective onto combinatorial proofs, and here we extract a syntactically motivated closure of sequent calculus from which there is a surjection onto a complete set of combinatorial proofs. We characterize a class of canonical sequent calculus proofs for the full set of propositional tautologies and derive a new completeness theorem for combinatorial propositions. For this, we define a new mapping between combinatorial proofs and sequent calculus proofs, different from the one originally proposed, which explicitly links the logical flow graph of a proof to a skew fibration between graphs of formulas. The categorical properties relating the original and the new mappings are explicitly discussed.

1. Introduction

The notion of a *formula* is usually associated to a sequential string of symbols or to a tree-like structure. But a formula can be represented as a more complex geometrical object, like a graph, a 2-cell, a polyhedra, or a circuit for instance, and its satisfiability can be characterized to be a geometrical property. The idea of using graphs to represent logical formulas has being exploited to show the *NP*-completeness of several graph-theoretical problems, as finding a "clique" or a "set-covering" [18]. Here we analyze another graph-theoretical representation of formulas, which, in contrast with the ones used in computational complexity, is used to characterize validity and could be generalized to arbitrary predicate formulas. The idea is to consider propositional formulas and proofs as colored graphs and define an embedding of formulas into proofs. The embedding can be intuitively thought to be a "projection" of the formula in its proof. For certain formulas, the geometrical characterization happens to suggest the structure of the proof for the formula and both the formula and its proof are associated to the same colored graph [16]. These formulas are provable in multiplicative linear logic with mix and for this logical system, a graph-theoretical property guarantees a graph representing a formula to be a graph of a proof. Namely, the cograph of a formula is a proof whenever any alternate elementary cycle in it contains a chord, and viceversa [16]. This beautiful structural result proved for a fragment of linear logic, does not hold for classical proofs because of the collapse of vertices in the graph associated to contractions. Hence, the identification of a purely geometrical criterium to guarantee that a graph is a graph for a classical proof becomes difficult but intriguing. An approach to investigate the geometrical complexity of this question is proposed in [12] where the logical language of formulas and proofs is replaced by a purely combinatorial language of graphs and homomorphisms. The graph-theoretical representation of propositional formulas as colored cographs (named combinatorial propositions in [12]) is considered, and the novel notion of combinatorial proof is introduced. Intuitively, a proof is defined to be a homomorphism (lax form of fibration) between the cograph representing the axioms of the proof and the cograph representing the provable formula. A completeness and soundness for combinatorial propositions is proved, that is the formula B is true if and only if B has a combinatorial proof [12]. The explicit link between combinatorial proofs and

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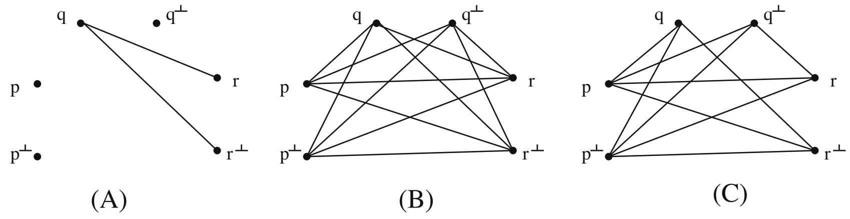

Fig. 1. Graph (A) corresponds to the formula $(p \vee q^{\perp}) \vee ((q \wedge (r \vee r^{\perp})) \vee p^{\perp})$, (B) is the graph of $(p \vee p^{\perp}) \wedge ((q \vee q^{\perp}) \wedge (r \vee r^{\perp}))$ and (C) is the graph of $(p \vee p^{\perp}) \wedge ((r \wedge q^{\perp}) \vee (q \wedge r^{\perp}))$.

sequent calculus proofs has been addressed in [13], where a surjective map from a semantically motivated closure of sequent calculus onto combinatorial proofs is provided. In this paper, we define another mapping for all combinatorial propositions, which is different from the one proposed in [13] and it might be considered intuitively closer to [16] (see also [14]). Namely, we construct a mapping (called G) of combinatorial proofs of classical propositional tautologies into sequent calculus (LK-) proofs and viceversa (called F). Our map G shows that logical paths passing through formulas in LK-proofs [2,5,9] explicitly define skew fibrations between graphs of formulas. G allows the definition of a class of "canonical" LK-proofs X_{can} such that $FG(X_{can}) = X_{can}$, and the derivation of a new proof of soundness and completeness for the combinatorial proof system.

2. Some basic definitions: formulas as cographs

In this section we associate formulas to cographs. We start with the observation that the connective $\wedge$ creates a tighter semantical link between two formulas than the connective $\vee$ and that the tree-like representation of a formula is not sufficient to capture this fact. A bit more structure has to be added to the representation.

Graphs associated to formulas. Without loss of generality, we allow negations to act on atomic formulas only, and we denote the formula $\neg A$ with the symbol $A^{\perp}$. Let us call *atomic* all occurrences of a propositional variable in a formula as well as all occurrences of its negation. A formula A, of arbitrary logical complexity, is associated to a graph G_A as follows: the vertices of G_A are all the atomic *occurrences* in A; the edges of G_A are defined by induction on the subformulas of A as follows:

1. If A is of the form $C \wedge B$ then G_A is obtained by adding regular edges between any vertex in G_C and any vertex in G_B.
2. No other edge appears in the graph.

Some examples of graph associated to formulas are given in Fig. 1. In graph (A), there are two edges between q and $r, r^{\perp}$ and they represent the fact that the variable q is linked with a connective $\wedge$ to the subformula $r \vee r^{\perp}$ in the formula. No other regular edge has been drawn since no other $\wedge$ is present in the formula. For the figures (B) and (C) the construction is done following the same idea.

Proposition 1. *Given a formula A there is a unique graph G_A associated to it.*

Proof. By induction on the complexity of the formula A, the only interesting case is the treatment of the logical connective $\wedge$. Suppose that $\wedge$ is applied to two subformulas B and C. By induction we can construct two graphs G_B, G_C which are uniquely associated to B and C. By definition we construct G_A through a matching between G_B and G_C which links each node of G_B to all the nodes of G_C. The links involve *all* nodes of G_B and of G_C. This ensures the uniqueness of the graph G_A. □

Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. An edge connecting the vertices x and y of G is denoted xy. The vertices x, y of xy are called *extremes* of the edge.

Definition 2. The class of *cographs* is the smallest class of simple graphs containing all one-vertex graphs, and closed under the two following operations:
1. *Complement*: $(V, E)^c = (V, E^c)$, where for all $x, y \in V$, $xy \in E^c$ iff $xy \notin E$.
2. *Disjoint union*: $(V, E) \oplus (V', E') = (V \uplus V', E \uplus E')$.

As reported in Mohring's survey [15], the following old observation rediscovered many times can be shown.

Proposition 3. *Let $G = (V, R)$ be a graph. G is a cograph if and only if the restriction of R to four vertices x, y, z, w never is the graph whose edges are xy, yz, zw.*

and based on it, one easily derives.