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A sketch of a dynamic epistemic semiring

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ABSTRACT

This paper proposes a semiring formulation for reasoning about an agent's changing beliefs: a *dynamic epistemic semiring* (DES). A DES is a modal semiring extended with epistemic-action operators. The paper concentrates on the revision operator by proposing an axiomatisation, developing a basic calculus and deriving the classical AGM revision axioms in the algebra. Iterated action is also considered.

1. Introduction

Formal reasoning about epistemic and doxastic notions in the 20th century can be traced back to Georg Henrik von Wright's *An Essay in Modal Logic* [30] and Hintikka's *Knowledge and Belief* [16]. These books were written as attempts to philosophically clarify our notions of knowledge and belief using modal logic. In the works of von Wright and Hintikka, the ones believing or knowing – the agents – cannot change their beliefs, *i.e.* only static aspects of an agent's beliefs can be reasoned about. Alchourrón et al. presented a semiformal framework for reasoning about an agent's changing beliefs [2] – this constituted the prelude for the vast amount of research on belief revision done in the last two decades. Segerberg was one of the first to make these streams flow together. In an array of papers (see, for example, [24,25,26]) he proposes and develops a fully formal framework for reasoning about an agent's changing beliefs: dynamic doxastic logic. Dynamic doxastic logic is, as the name discerns, a blend of dynamic and doxastic logic. In more recent times, formal reasoning about knowledge has been put to use in computer science, the standard references seem to be [15,18,29,14]. (The epithet epistemic is usually used in computer science, whereas in philosophical logic one tends to distinguish between epistemic and doxastic. We will henceforth use 'epistemic', except when referring to Segerberg's logic.)

Since Kozen published his axiomatisation of Kleene algebra (an idempotent semiring with the Kleene star as a least fixpoint with respect to the canonical order) [19] and also showed how to use an elaboration of it for program reasoning [20], there has been significant development and application of something that could be called semiring structures (some form of semiring equipped with additional operators). The spirit in this development lies very much in the calculational prospect of abstract algebra – tedious model-theoretic reasoning can be reduced to simple, perspicuous, symbol-pushing calculations, and, in addition, the level of abstraction also makes mechanisation and automation tractable [1,17]. One important development are the modal semirings by Desharnais et al. [12]. A modal semiring is a semiring structure including a domain operator facilitating the definition of modal operators in the sense of dynamic logic.

Our intent in this paper is to let yet another stream run up to Segerberg's uniting work by viewing some aspects of dynamic doxastic logic from the point of modal semirings. In this paper, we propose a modal semiring extended with an epistemic-action operator: a *dynamic epistemic semiring*. This structure allows us to reason about an agent's changing beliefs in a transparent, calculational fashion. The carrier elements of the algebra are viewed as epistemic actions – actions working on the agent's beliefs. To check whether the agent believes a proposition we introduce special actions: *epistemic tests*. Epistemic tests work like guards in program theory, *i.e.* programs that check if some predicate holds or not.

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In comparison to the semi-formal approach of the AGM trio, and the logical approaches of Segerberg [24,25,26]; Baltag et al. [6,5] and the so called Amsterdam school centered around van Benthem [8] and de Rijke [23] – our approach is abstract-algebraic. The work that comes closest to ours is the elegant algebraic approach to dynamic epistemic reasoning by Baltag et al. [3,4,7]. But when they use systems (a combination of a complete sup-lattice, a quantale and a right-module structure on the lattice) as the underlying structure in order to achieve an axiomatisation, we use modal semirings. Baltag et al.'s approach is perhaps more thoroughly developed to match the original framework by Alchourrón et al. than ours, whereas our treatment, on the other hand, is more perspicuous in that it has a (at least to our mind) simpler algebraic structure underneath, and it also easily facilitates iterated action.

The structure of the paper is: Section 2 defines modal semirings. In Section 3, the revision operator is introduced. In Section 4, we provide a relational model and in Section 5, we develop a basic calculus. In Section 6, we derive the classical AGM axioms for revision in the algebra. Section 7 considers an extension with an iteration operator. The final section contains some concluding remarks. Only the most interesting proofs are included in the main text; Appendix A contains detailed proofs of the remaining claims.

We want to emphasise that the main purpose of *this* paper is not to consider applications or the philosophy of formal epistemology, but to lay an initial foundation to using semirings for reasoning about belief in change. The paper, then, primarily provides a concise technical framework, that connects earlier work on formal epistemic reasoning with the recent work on applications of semirings. It is an invited extension of an earlier paper [28].

2. Modal semirings

By an *idempotent semiring* we shall understand a structure over the signature $(+, ;, 0, 1)$ such that the reduct over $(+, 0)$ is a commutative and idempotent monoid, and the reduct over $(;, 1)$ is a monoid such that $;$ distributes over $+$ and has 0 as a left and right annihilator. When no risk for confusion arises $;$ will be left implicit. An idempotent semiring thus satisfies the following axioms (a, b and c in the carrier set):

$$a + (b + c) = (a + b) + c, \tag{1}$$
$$a + b = b + a, \tag{2}$$
$$a + 0 = a, \tag{3}$$
$$a + a = a, \tag{4}$$
$$a(bc) = (ab)c, \tag{5}$$
$$1a = a = a1, \tag{6}$$
$$0a = 0 = a0, \tag{7}$$
$$a(b + c) = ab + ac \text{ and} \tag{8}$$
$$(a + b)c = ac + bc. \tag{9}$$

We define the *canonical order* $\leq$ on a semiring by $a \leq b \Leftrightarrow_{df} a + b = b$ for all a and b in the carrier set. With respect to the canonical order, 0 is the least element, $;$ as well as $+$ are isotone and $+$ is join.

A *test semiring* [11,20] is a two-sorted algebra

$$(S, \mathsf{test}(S), +, ;, \neg, 0, 1)$$

such that

- $(S, +, ;, 0, 1)$ is an idempotent semiring,
- $(\mathsf{test}(S), +, ;, \neg, 0, 1)$ is a Boolean algebra (BA) and
- $\mathsf{test}(S) \subseteq S$.

Join and meet in $\mathsf{test}(S)$ are thus $+$ and $;$, respectively, and the complement is denoted by $\neg$; 0 is the least and 1 is the greatest element. We shall use $a, b, \ldots$ for general semiring elements and $p, q, \ldots$ for test elements. On a test semiring we axiomatise a *domain operator* $\ulcorner : S \to \mathsf{test}(S)$ by

$$a \leq \ulcorner a ; a, \tag{10}$$
$$\ulcorner(pa) \leq p \text{ and} \tag{11}$$
$$\ulcorner(a\ulcorner b) \leq \ulcorner(ab), \tag{12}$$

for all $a \in S$ and $p \in \mathsf{test}(S)$ [11]. Inequalities (10) and (12) can be strengthened to equalities. To be explicit, the operator $\ulcorner$ binds the strongest, followed in descending order by $\cdot$ and $+$.

The domain operator satisfies stability of tests,

$$\ulcorner p = p, \tag{13}$$