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Parameterized complexity of the anchored k-core problem for directed graphs ☆

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ABSTRACT

We consider the Directed Anchored k-Core problem, where the task is for a given directed graph G and integers b, k and p, to find an induced subgraph H with at least p vertices (the core) such that all but at most b vertices (the anchors) of H have in-degree at least k. We undertake a systematic analysis of the computational complexity of the Directed Anchored k-Core problem.

1. Introduction

Degree-constrained subgraph problems have been extensively studied in theoretical computer science. One can describe degree-constrained subgraph problems in the following general setting: given a (un)directed graph G, find a maximum/minimum sized (induced, connected) subgraph H subject to some condition $\mathcal{C}$ imposed on the degrees of vertices. For example, Independent Set or (Induced) Matching can be seen as problems within this framework. In this paper, we study an interesting variant of the degree-constrained subgraph problem where we have to find a large subgraph in which all (except a small set of anchor vertices) satisfy a degree constraint. Such problems arise in different settings in social sciences. Adding the anchors however leads to non-trivial computation challenges as we will see in this paper.

More precisely, the k-core of a directed graph G is defined as the largest subgraph H such that $\deg^-_H(v) \geq k$ for every $v \in V(H)$. This notion was introduced by Seidman [17] and is a well-known concept in the theory of social networks. It has also been studied in various social sciences literature [8,9]. It is easy to see that we can find the k-core of a given directed graph in polynomial time by the following procedure: iteratively remove any vertex that has in-degree less than k. However, one might not want to strictly enforce the condition of in-degree being at least k for every vertex. In particular, we allow for a small number of special vertices (called anchors) which can have arbitrary in-degrees, but their purpose in the (anchored) k-core is to augment the in-degrees of the non-anchored vertices. Bhawalkar et al. [2] introduced the Anchored k-Core problem for (undirected) graphs. In the Anchored k-Core problem the input is an undirected graph $G = (V, E)$ and integers b, k, and the task is to find an induced subgraph H of maximum size with all vertices but at most b (which are

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anchored) to be of degree at least k. In this work, we extend the notion of anchored k-core to directed graphs and define the parameterized version of the problem formally:

> DIRECTED ANCHORED k-CORE (DIR-AKC)
> *Input*: A directed graph $G = (V, E)$ and integers b, k, p.
> *Parameter 1*: b.
> *Parameter 2*: k.
> *Parameter 3*: p.
> *Question*: Do there exist sets of vertices $A \subseteq U \subseteq V(G)$ such that $|A| \leq b$, $|U| \geq p$, and every $v \in U \setminus A$ satisfies $d^-_{G[U]}(v) \geq k$?

We will refer to the set A as the set of *anchors* and to the graph $H = G[U]$ as the *anchored k-core*. Note that the undirected version of the ANCHORED k-CORE problem can be modeled by the directed version: simply replace each edge $\{u, v\}$ by arcs (u, v) and (v, u). Keeping the parameters b, k, p unchanged it is now easy to see that the two instances are equivalent.

Connection to preventing unraveling in social networks Social networks are generally represented by making use of undirected or directed graphs, where the edge set represents the relationship between individuals in the network. The undirected graph model works fine for some networks, say Facebook, but the nature of interaction on some social networks such as Twitter is asymmetrical: the fact that user A follows user B does not imply that user B also follows A. In this case, it is more appropriate to model interactions in the network by *directed* graphs. We add a directed edge (u, v) if v follows u. We can consider a model of *user engagement* where there is a threshold value k, such that each individual with less than k people to follow (or equivalently whose in-degree is less than k) drops out of the network. This process can be contagious, and may affect even those individuals who initially were linked to more than k people. An extreme example of this was given by Schelling (see p. 17 of [15]): consider a directed path on n vertices and let $k = 1$. The left-endpoint has in-degree zero, it drops out and now the in-degree of its only out-neighbor in the path becomes zero and it drops out as well. It is not hard to see that this way the whole network eventually drops out as the result of a *cascade of iterated withdrawals*, i.e., the 1-core of this graph is the empty set. The unraveling process described above in Schelling's example of a directed path can be highly undesirable in many scenarios. One can attempt to prevent this unraveling by introducing a few special vertices (called anchors) by "buying" them with extra incentives.

Parameterized complexity We are mainly interested in the parameterized complexity of ANCHORED k-CORE. For general background on parameterized complexity, we refer to the recent books by Cygan et al. [10] and Downey and Fellows [12]. Parameterized complexity is basically a two dimensional framework for studying the computational complexity of a problem. One dimension is the input size n and another one is a parameter k. A problem is said to be *fixed parameter tractable* (or FPT) if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function f. A problem is said to be in XP, if it can be solved in time $O(n^{f(k)})$ for some function f. The W-hierarchy is a collection of computational complexity classes: we omit the technical definitions here. The following relation is known amongst the classes in the W-hierarchy: $\text{FPT} = \text{W}[0] \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \ldots \subseteq \text{W}[P] \subseteq \text{XP}$. It is widely believed that $\text{FPT} \neq \text{W}[1]$, and hence if a problem is hard for the class W[i] (for any $i \geq 1$) then it is considered to be fixed-parameter intractable.

Previous results for undirected graphs Bhawalkar et al. [2] initiated the algorithmic study of ANCHORED k-CORE on undirected graphs. In particular, they obtained the following dichotomy result: the decision version of the problem is solvable in polynomial time for $k \leq 2$ and is NP-complete for all $k \geq 3$. In a followup paper, the current set of authors showed that for $k \geq 3$ the problem remains NP-complete even on planar graphs [7]. This motivates the study of the problem for $k \geq 3$ from the viewpoint of parameterized complexity. Unfortunately, the problem is W[2]-hard parameterized by b [2] and W[1]-hard parameterized by p even for $k = 3$ [7].

Our results In this paper, we initiate the study of ANCHORED k-CORE on directed graphs and provide a new insight into the computational complexity of the problem. We obtain the following results.

- The decision version of DIR-AKC is NP-complete for every $k \geq 1$ even if the input graph is restricted to be a planar directed acyclic graph (DAG) of maximum degree at most $k + 2$. Thus the directed version is in some sense strictly harder than the undirected version which is known be in P if $k \leq 2$, and NP-complete if $k \geq 3$ [2]. These results are proven in Section 2.
- The NP-hardness result for DIR-AKC motivates us to make a more refined analysis of the DIR-AKC problem via the paradigm of parameterized complexity. We obtain (Section 3) the following dichotomy result: DIR-AKC is FPT parameterized by p if $k = 1$, and W[1]-hard if $k \geq 2$.

This fixed-parameter intractability result parameterized by p forces us to consider the complexity on special classes of graphs such as bounded-degree directed graphs or directed acyclic graphs.