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Distance constraint satisfaction problems ☆

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ABSTRACT

We study the complexity of constraint satisfaction problems for templates Γ over the integers where the relations are first-order definable from the successor function. In the case that Γ is locally finite (i.e., the Gaifman graph of Γ has finite degree), we show that Γ is homomorphically equivalent to a structure with one of two classes of polymorphisms (which we call modular max and modular min) and the CSP for Γ can be solved in polynomial time, or Γ is homomorphically equivalent to a finite transitive structure, or the CSP for Γ is NP-complete. Assuming a widely believed conjecture from finite domain constraint satisfaction (we require the *tractability conjecture* by Bulatov, Jeavons and Krokhin in the special case of *transitive* finite templates), this proves that those CSPs have a complexity dichotomy, that is, are either in P or NP-complete.

1. Introduction

Constraint satisfaction problems appear naturally in many areas of theoretical computer science, for example in artificial intelligence, optimization, computer algebra, computational biology, computational linguistics, and type systems for programming languages. Such problems are typically NP-hard, but sometimes they are polynomial-time tractable. The question as to which CSPs are in P and which are hard has stimulated a lot of research in the past 15 years. For pointers to the literature, there is a collection of survey articles [15].

The *constraint satisfaction problem CSP* for a fixed (not necessarily finite) structure Γ with a finite relational signature τ is the computational problem of deciding whether a given primitive positive sentence is true in Γ. A formula is *primitive positive* if it is of the form $\exists x_1, \ldots, x_n\, (\psi_1 \wedge \cdots \wedge \psi_m)$ where each ψ_i is an *atomic formula* over Γ, that is, a formula of the

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form $y_1 = y_2$ or $R(y_1, \ldots, y_j)$ for a relation symbol R of a relation from Γ. The structure Γ is also called the *template* of the CSP.

The class of problems that can be formulated as a CSP for a fixed structure Γ is very large. It can be shown that for every computational problem there is a structure Γ such that the CSP for Γ is equivalent to this problem under polynomial-time Turing reductions [5]. This makes it very unlikely that we can give good descriptions of all those Γ where the CSP for Γ is in P. In contrast, the class of CSPs for a *finite* structure Γ is quite restricted, and indeed it has been conjectured that the CSP for Γ is either in P or NP-complete in this case [17]. So it appears to be natural to study the CSP for classes of infinite structures Γ that share good properties with finite structures.

In graph theory and combinatorics, there are two major concepts of *finiteness* for infinite structures. The first is ω-categoricity: a countable structure is ω-categorical if and only if its automorphism group has for all n only finitely many orbits in its natural action on n-tuples [14,23,21]. This property has been exploited to transfer techniques that were known to analyze the computational complexity of CSPs with finite domains to infinite domains [10,7,11]; see also the introduction of [3].

The second concept of finiteness is the property of an infinite graph or structure to be *locally finite* (see Section 8 in [16]). A graph is called locally finite if every vertex is contained in a finite number of edges; a relational structure is called locally finite if its Gaifman graph (definition given in Section 2) is locally finite. Many conjectures that are open for general infinite graphs become true for locally finite graphs, and many results that are difficult become easy for locally finite graphs.

In this paper, we initiate the study of CSPs with locally finite templates by studying locally finite templates Γ that have a *first-order definition* in $(\mathbb{Z}; \mathrm{succ})$, that is, Γ has the domain $\mathbb{Z}$ and all relations of Γ can be defined by a first-order formula over the successor relation on the integers, $\mathrm{succ} = \{(x, y) \mid y = x + 1\}$.

As an example, consider the directed graph with vertex set $\mathbb{Z}$ which has an edge between x and y if the difference, $y - x$, between x and y is either 1 or 3. This graph is the structure $(\mathbb{Z}; \mathrm{Diff}_{\{1,3\}})$ where $\mathrm{Diff}_{\{1,3\}} = \{(x, y) \mid y - x \in \{1, 3\}\}$, which has a first-order definition over $(\mathbb{Z}; \mathrm{succ})$ since $\mathrm{Diff}_{\{1,3\}}(x, y)$ if and only if

$$\mathrm{succ}(x, y) \vee \exists u, v\, (\mathrm{succ}(x, u) \wedge \mathrm{succ}(u, v) \wedge \mathrm{succ}(v, y)).$$

Another example is the undirected graph $(\mathbb{Z}; \mathrm{Dist}_{\{1,2\}})$ with vertex set $\mathbb{Z}$ where two integers x, y are linked in $\mathrm{Dist}_{\{1,2\}}$ if the *distance*, $|y - x|$, is one or two.

Structures with a first-order definition in $(\mathbb{Z}; \mathrm{succ})$ are particularly well-behaved from a model-theoretic perspective: all of those structures are strongly minimal [23,21], and therefore uncountably categorical. Uncountable models of their first-order theory will be saturated; for implications of those properties for the study of the CSP, see [6]. In some sense, $(\mathbb{Z}; \mathrm{succ})$ constitutes one of the simplest infinite structures that is not ω-categorical.

The corresponding class of CSPs contains many natural combinatorial problems. For instance, the CSP for the structure $(\mathbb{Z}; \mathrm{Diff}_{\{1,3\}})$ is the computational problem of labeling the vertices of a given finite directed graph G such that if (x, y) is an arc in G, then the difference between the label for y and the label for x is one or three. It follows from our general results that this problem is in P. The CSP for the undirected graph $(\mathbb{Z}; \mathrm{Dist}_{\{1,2\}})$ is exactly the 3-coloring problem, and thus NP-complete. This is readily seen if one observes that any homomorphism of a graph G into the template modulo 3 gives rise to a 3-coloring of G. In general, the problems that we study in this paper have the flavor of assignment problems where we have to assign integers to variables such that various given constraints on differences and distances (and Boolean combinations thereof) between variables are satisfied. We therefore call the class of CSPs whose template is locally finite and definable over $(\mathbb{Z}; \mathrm{succ})$ *distance CSPs*. Our main result is the following classification result for distance CSPs.

Theorem 1. *Let Γ be a locally finite structure with a first-order definition in $(\mathbb{Z}; \mathrm{succ})$. Then at least one of the following applies.*

- *Γ has an endomorphism with finite range, and the CSP for Γ equals the CSP for a finite structure;*
- *the CSP for Γ is NP-complete;*
- *Γ is homomorphically equivalent to a structure with a first-order definition in $(\mathbb{Z}; \mathrm{succ})$ which has a binary modular max or modular min polymorphism, and the CSP for Γ is in P.*

If a locally finite structure Γ with a first-order definition in $(\mathbb{Z}; \mathrm{succ})$ has a finite core, then a widely accepted conjecture about finite domain CSPs implies that the CSP for Γ is either NP-complete or in P. In fact, for this we only need the (open) special case of the conjecture of Feder and Vardi [17] that states that the CSP for finite templates with a transitive automorphism group is either in P or NP-complete (see Section 7 for details).

To show our theorem, we prove that if the first two items of the statement do not apply, then Γ is homomorphically equivalent to a structure Δ with a first-order definition in $(\mathbb{Z}; \mathrm{succ})$ that has one of two specific classes of polymorphism which we call *modular max* and *modular min* (defined in Section 5). Using these polymorphisms, we further show that the CSP for Δ, and hence also that for Γ, can be solved in polynomial time by certain arc consistency techniques. Polynomial-time tractability results based on arc consistency were previously known for finite or ω-categorical templates; using the local finiteness assumption we manage to apply such techniques to templates which are not ω-categorical.

On the way to our classification result we derive several facts about structures definable in $(\mathbb{Z}; \mathrm{succ})$, and automorphisms and endomorphisms of these structures, which might be of independent interest in model theory, universal algebra, and