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Existential second-order logic and modal logic with quantified accessibility relations

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A R T I C L E I N F O

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A B S T R A C T

This article investigates the role of *arity* of second-order quantifiers in existential second-order logic, also known as Σ^1_1. We identify fragments L of Σ^1_1 where second-order quantification of relations of arity $k > 1$ is (nontrivially) vacuous in the sense that each formula of L can be translated to a formula of (a fragment of) *monadic* Σ^1_1. Let *polyadic Boolean modal logic with identity* ($\mathrm{PBML}^=$) be the logic obtained by extending standard polyadic multimodal logic with built-in identity modalities and with constructors that allow for the Boolean combination of accessibility relations. Let $\Sigma^1_1(\mathrm{PBML}^=)$ be the extension of $\mathrm{PBML}^=$ with existential prenex quantification of accessibility relations and proposition symbols. The principal result of the article is that $\Sigma^1_1(\mathrm{PBML}^=)$ translates into *monadic* Σ^1_1. As a corollary, we obtain a variety of decidability results for multimodal logic. The translation can also be seen as a step towards establishing whether every property of finite directed graphs expressible in $\Sigma^1_1(\mathrm{FO}^2)$ is also expressible in monadic Σ^1_1. This question was left open in the 1999 paper of Grädel and Rosen in the 14th Annual IEEE Symposium on Logic in Computer Science.

1. Introduction

Properties of existential second-order logic have been widely studied in finite model theory. Existential second-order logic captures the complexity class NP, and there exists a large body of results concerning the expressive power of different fragments of the logic (see e.g. [1,6,14,16,21]). However, there are several issues related to the expressivity of Σ^1_1 that are not understood well. Most notably, Fagin's *spectrum arity hierarchy conjecture* (see [8,9]) remains a longstanding difficult open problem in finite model theory. Fagin's question is whether there exist sets of positive integers (spectra) definable by first-order sentences[2] with predicates of maximum arity $k+1$, but not definable by sentences with predicates of arity k.

In this article we investigate *arity reduction* of formulae of existential second-order logic: we identify fragments L of Σ^1_1 where second-order quantification of relations of arity $k > 1$ is (nontrivially) vacuous in the sense that each formula of L can be translated to a formula of (a fragment of) monadic Σ^1_1, also known as ∃MSO. Our work is directly related to a novel perspective on *modal correspondence theory*, and our investigations lead to a variety of *decidability results* concerning multimodal logics over classes of frames with built-in relations. Our work also aims to provide a stepping stone towards a solution of an open problem of Grädel and Rosen posed in [15].

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[2] The spectrum of a sentence φ is the set of positive integers n such that φ has a model of the size n.

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The objective of modal correspondence theory (see [3]) is to classify formulae of modal logic according to whether they define elementary classes of Kripke frames.[3] On the level of frames, modal logic can be considered a fragment of monadic Π^1_1, also known as ∀MSO, and therefore correspondence theory studies a special fragment of ∀MSO.

When a modal formula is inspected from the point of view of Kripke frames, the *proposition symbols* occurring in the formula are quantified universally; it is natural to ask what happens if one also quantifies some of the *binary relation symbols* occurring in (the standard translation of) a modal formula. This question is investigated in [22], where the focus is on the expressivity of multimodal logic with universal prenex quantification of (some of) the binary and unary relation symbols occurring in a formula. A question that immediately suggests itself is whether there exists any class of multimodal frames definable in this logic, let us call it $\Pi^1_1(\mathrm{ML})$, but not definable in monadic second-order logic MSO. The question can be regarded as a question of modal correspondence theory. Here, however, the correspondence language is MSO rather than FO. For further investigations that involve quantification of binary relations in modal logic, see for example [4,23].

In the current article we investigate two multimodal logics with existential second-order prenex quantification of accessibility relations and proposition symbols, $\Sigma^1_1(\mathrm{PBML}^=)$ and $\Sigma^1_1(\mathrm{ML})$. The logic $\Sigma^1_1(\mathrm{ML})$ is the extension of ordinary multimodal logic with existential second-order prenex quantification of binary accessibility relations and proposition symbols. $\mathrm{PBML}^=$ is the logic obtained by extending standard polyadic[4] multimodal logic by built-in identity modalities and by constructors that allow for the Boolean combination of accessibility relations (see Subsection 2.1). Obviously $\Sigma^1_1(\mathrm{PBML}^=)$ is the extension of $\mathrm{PBML}^=$ with existential second-order prenex quantification of accessibility relations and proposition symbols.

We warm up by showing that $\Sigma^1_1(\mathrm{ML})$ translates into *monadic* $\Sigma^1_1(\mathrm{MLE})$, which is the extension of multimodal logic with the global modality and existential second-order prenex quantification of *only* proposition symbols. The method of proof is based on the notion of a *largest filtration* (see [3] for the definition). We then push the method and show that $\Sigma^1_1(\mathrm{PBML}^=)$ translates into monadic Σ^1_1. Note that both of these results immediately imply that $\Pi^1_1(\mathrm{ML})$ translates into ∀MSO and therefore show that MSO would be a somewhat dull correspondence language for correspondence theory of $\Pi^1_1(\mathrm{ML})$.

The logic $\mathrm{PBML}^=$ contains a wide variety of logics used in different applications of modal logic. It could be argued that $\{\neg, \cup, \cap, \circ, ^*, ^\smile, E, D\}$ is more or less the core collection of operations on binary relations used in extensions of modal logic defined for the purposes of applications. Here $\neg$, $\cup$, $\cap$, $\circ$, *, $^\smile$ denote the complement, union, intersection, composition, transitive reflexive closure and converse operations, respectively. The symbols E and D denote the global modality and difference modality. Logics using some of these core operations include for example propositional dynamic logic PDL [10,17] and its extensions, Boolean modal logic [11,25], description logics [2,19,27], modal logic with the global modality [13] and modal logic with the difference modality [5]. The operations $\neg, \cup, \cap, E, D$ are part of $\mathrm{PBML}^=$. One of our principal motivations for studying $\mathrm{PBML}^=$ is that the logic subsumes a large number of typical extensions of modal logic. Our translation from $\Sigma^1_1(\mathrm{PBML}^=)$ into ∃MSO gives as a direct corollary a wide range of decidability results for extensions of multimodal logic over various classes of Kripke frames with built-in relations; see Theorem 4.10 below.

In addition to applied modal logics, the investigations in this article are directly related to an interesting open problem concerning two-variable logics. Grädel and Rosen ask in [15] the question whether there exists any class of finite directed graphs that is definable in $\Sigma^1_1(\mathrm{FO}^2)$ but not in ∃MSO. Let $\mathrm{BML}^=$ denote ordinary Boolean modal logic with a built-in identity relation, i.e., $\mathrm{BML}^=$ is the restriction of $\mathrm{PBML}^=$ to binary relations. Lutz, Sattler and Wolter show in the article [26] that $\mathrm{BML}^=$ extended with the converse operator is expressively complete for FO^2. Therefore, in order to prove that $\Sigma^1_1(\mathrm{FO}^2) \leq \exists\mathrm{MSO}$, one would have to modify our translation from $\Sigma^1_1(\mathrm{BML}^=)$ into ∃MSO such that it takes into account the possibility of using the converse operation. We have succeeded neither in this nor in identifying a $\Sigma^1_1(\mathrm{FO}^2)$ definable class of directed graphs that is not definable in ∃MSO. However, we find modal logic a promising framework for working on the problem.

This article is the journal version of the conference paper [18].

2. Preliminary definitions

In this section we discuss technical notions that occupy a central role in the rest of the article.

2.1. Syntax and semantics of $\Sigma^1_1(\mathrm{PBML}^=)$

The semantics of $\mathrm{PBML}^=$—defined in detail below—is obtained by combining the semantics of Boolean modal logic with the standard generalization of Kripke semantics to polyadic modal contexts. Intuitively, Boolean modal logic is simply multimodal logic where new modalities can be syntactically defined by forming Boolean combinations of accessibility relations.

Let V be a vocabulary containing relation symbols only. A *V-model*, or a *model of the vocabulary V*, is an ordinary first-order model (see [7]) that gives an interpretation to exactly all the symbols in V. We use this notion of a model in both

[3] It is well known that if a class of Kripke frames is definable by a modal formula, then the class is definable by a set of FO formulae iff it is definable by a single FO formula. See [12] for example. Therefore it makes no difference *here* whether the term “elementary” is taken to mean definability by a single first-order formula or definability by a set of first-order formulae.

[4] Modal logics with accessibility relations of arities greater than two are called polyadic. See Section 2.1 for the related definitions.