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On the isomorphism problem for Helly circular-arc graphs

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ABSTRACT

The isomorphism problem is known to be efficiently solvable for interval graphs, while for the larger class of circular-arc graphs its complexity status stays open. We consider the intermediate class of intersection graphs for families of circular arcs that satisfy the Helly property. We solve the isomorphism problem for this class in logarithmic space. If an input graph has a Helly circular-arc model, then our algorithm constructs it canonically, which means that the models constructed for isomorphic graphs are equal.

1. Introduction

An *intersection representation* of a graph G is a mapping α from the vertex set $V(G)$ onto a family $\mathcal{A}$ of sets such that vertices u and v of G are adjacent if and only if the sets $\alpha(u)$ and $\alpha(v)$ have a nonempty intersection. The family $\mathcal{A}$ is called an *intersection model* of G. G is an *interval graph* if it admits an intersection model consisting of intervals of reals (or, equivalently, intervals of consecutive integers). The larger class of *circular-arc (CA) graphs* arises if we consider intersection models consisting of arcs on a circle. These two archetypal classes of intersection graphs have important applications, most noticeably in computational genomics, and have been intensively studied for decades in graph theory and algorithmics; for an overview see e.g. [21]. In general, fixing a class of admissible intersection models, we obtain the corresponding class of intersection graphs.

In the *canonical representation problem* for a class $\mathcal{C}$ of intersection graphs, we are given a graph $G \in \mathcal{C}$ and have to compute its intersection representation α so that isomorphic graphs receive equal intersection models. This subsumes both the recognition of $\mathcal{C}$ and the isomorphism testing for graphs in $\mathcal{C}$. In their seminal work [1,16], Booth and Lueker solve both the representation and the isomorphism problems for interval graphs in linear time. Together with Laubner, we designed a canonical representation algorithm for interval graphs that takes logarithmic space [11].

The case of CA graphs remains a challenge up to now. While a circular-arc intersection model can be constructed in linear time (McConnell [17]), no polynomial-time isomorphism test for CA graphs is currently known (though some approaches [8] have appeared in the literature; see the discussion in [4]). A few natural subclasses of CA graphs have received special attention among researchers. In particular, for proper CA graphs both the recognition and the isomorphism problems are solved in linear time, respectively, in [5,10] and [15,4], and in logarithmic space in [12]. The latter result actually gives a

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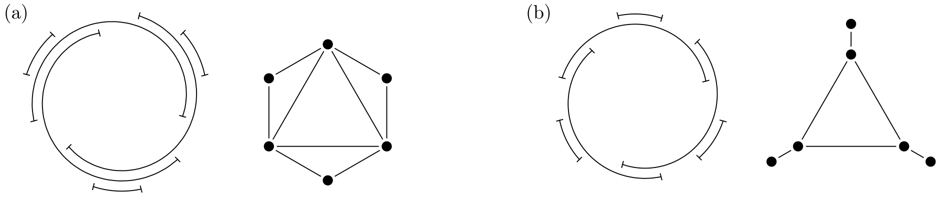

Fig. 1. Two non-Helly CA models and their intersection graphs. The graph in (a) admits an HCA model, while the graph in (b) does not.

logspace algorithm for computing a canonical representation of proper CA graphs, and such an algorithm is also known for unit CA graphs [20]. The history of the isomorphism problem for circular-arc graphs is surveyed in more detail by Uehara [24].

Here we are interested in the class of *Helly circular-arc (HCA) graphs*. Those are graphs that admit circular-arc models having the *Helly property*, which requires that every subfamily of arcs with nonempty pairwise intersections has a nonempty overall intersection. Obeying this property is assumed in the representation problem for HCA graphs. Since any family of intervals has the Helly property and the cycles of length at least 4 are HCA graphs but not interval, the canonical representation problem for HCA graphs generalizes the canonical representation problem for interval graphs. On the other hand, not every CA model is Helly; see Fig. 1 for examples. Joeris et al. characterize HCA graphs among CA graphs by a family of forbidden induced subgraphs [9].

HCA graphs were introduced by Gavril under the name of Θ circular-arc graphs [7]. Gavril gave an $O(n^3)$ time representation algorithm for HCA graphs. Hsu improved this to $O(nm)$ [8]. Recently, Joeris et al. gave a linear time representation algorithm [9]. The fastest known isomorphism algorithm for HCA graphs is due to Curtis et al. and works in linear time [4]. Chen gave a parallel AC^2 algorithm [2].

We aim at designing space efficient algorithms. In [14] we already presented a logspace canonical representation algorithm for HCA graphs. Our approach in [14] uses techniques developed by McConnell in [17], and the algorithm is rather intricate. Now we suggest an alternative approach that is independent of [17]. The new algorithm admits a much simpler analysis and exploits some new ideas that may be of independent interest.

Theorem 1.1. *The canonical representation problem for the class of Helly circular-arc graphs is solvable in logspace.*

Note that solvability in logspace implies solvability in logarithmic time by a CRCW PRAM with polynomially many parallel processors, i.e., in AC^1. Prior to our work, no AC^1 algorithm was known for recognition and isomorphism testing of HCA graphs.

In general, solvability of the isomorphism problem for a non-trivial class of graphs in logarithmic space is an interesting result because the general graph isomorphism problem is known to be DET-hard [22] and therefore also NL-hard. It is also interesting that for some classes of intersection graphs, the isomorphism problem is as hard as in general. For example, Uehara [23] shows this for intersection graphs of axis-parallel rectangles in the plane. Note that any family of such rectangles has the Helly property.

Our strategy Recall that a hypergraph $\mathcal{H}$ is *interval* (resp. *circular-arc*) if it is isomorphic to a hypergraph whose hyperedges are intervals of integers (resp. arcs of a discrete circle). Such an isomorphism is called an interval (resp. arc) representation of $\mathcal{H}$. Like in our approach to interval graphs in [11], the overall idea of our algorithm is to exploit the relationship between an input graph G and the dual of its clique hypergraph, which will be denoted by $\mathcal{B}(G)$. Fulkerson and Gross [6] established that G is an interval graph if and only if $\mathcal{B}(G)$ is an interval hypergraph. Moreover, represented as an interval system, $\mathcal{B}(G)$ can serve as an interval model of G. More specifically, our approach in [11] consists of two steps: First, construct $\mathcal{B}(G)$ (or, equivalently, find all maxcliques in G) and, second, design a canonical representation algorithm for interval *hypergraphs* and apply it to $\mathcal{B}(G)$. The first step is implementable in logspace because all interval graphs without isolated vertices are *maximal clique irreducible* [19]. This means that every maxclique C contains an edge uv that is contained in no other maxclique, and implies that C is equal to the common neighborhood of u and v (cf. Lemma 7.2).

The Fulkerson–Gross theorem is extended to the class of HCA graphs by Gavril [7]: G is an HCA graph if and only if $\mathcal{B}(G)$ is a CA hypergraph. Also in this case, any arc representation of $\mathcal{B}(G)$ yields a Helly arc representation for G. The canonical representation problem for CA hypergraphs is solved in logspace in [12]. However, the similarity between interval and HCA graphs ends there because HCA graphs are in general not maximal clique irreducible and hence we have to use a different approach to compute the hypergraph $\mathcal{B}(G)$.

Though we are not able to find all maxcliques of an HCA graph G directly, the discussion above shows that the *canonical representation* problem for HCA graphs is logspace reducible to the *representation* problem, where we need just to construct a Helly arc representation and do not need to take care of canonicity. Indeed, once we have an arbitrary HCA model of an input graph G, we get all maxcliques of G by inspection of the sets of arcs sharing a common point. As soon as all the maxcliques are found, we form the hypergraph $\mathcal{B}(G)$ and compute its canonical representation according to [12] (the details are given in Section 4).