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Initial segment complexities of randomness notions

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ABSTRACT

Schnorr famously proved that Martin-Löf-randomness of a sequence A can be characterised via the complexity of A's initial segments. Nies, Stephan and Terwijn as well as independently Miller showed that a set is 2-random (that is, Martin-Löf random relative to the halting problem K) iff there is no function f such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n - m$; before the proof of this equivalence the notion defined via the latter condition was known as Kolmogorov random.

In the present work it is shown that characterisations of this style can also be given for other randomness criteria like strong randomness (also known as weak 2-randomness), Kurtz randomness relative to K, Martin-Löf randomness of PA-incomplete sets, and strong Kurtz randomness; here one does not just quantify over all functions f but over functions f of a specific form. For example, A is Martin-Löf random and PA-incomplete iff there is no A-recursive function f such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n - m$. The characterisation for strong randomness relates to functions which are the concatenation of an A-recursive function executed after a K-recursive function; this solves an open problem of Nies.

In addition to this, characterisations of a similar style are also given for Demuth randomness, weak Demuth randomness and Schnorr randomness relative to K. Although the unrelativised versions of Kurtz randomness and Schnorr randomness do not admit such a characterisation in terms of plain Kolmogorov complexity, Bienvenu and Merkle gave one in terms of Kolmogorov complexity defined by computable machines.

1. Introduction

Kolmogorov complexity [13,18] aims to describe when a set is random in an algorithmic way. Here randomness means that no type of patterns can be exploited by an algorithm in order to generate initial segments of the characteristic function from shorter programs. Randomness notions have been formalised by Martin-Löf [14], Schnorr [23] and others. A special emphasis was put on describing randomness of a set A in terms of the complexity of the initial segments $A(0)A(1)\dots A(n)$. The first important result in that direction was that Schnorr [24] proved that a set A is Martin-Löf random if and only if for

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almost all n the prefix free Kolmogorov complexity $H(A(0)A(1)\dots A(n))$ of the $(n+1)$-th initial segment is at least n. It is easy to see that the counterpart of this characterisation is that a set A is *not Martin-Löf random* iff there is an A-recursive function f such that $H(A(0)A(1)\dots A(f(m))) \leqslant f(m) - m$ for all m. In other words, one can find – relative to A – points to witness the non-randomness effectively. It should be noted that the function f has to be taken relative to A and not relative to some fixed oracle B independent of A as the sets 2-generic relative to B are not Martin-Löf random but would not admit a B-recursive function f witnessing the non-randomness in the way just mentioned.

The scope of the present paper is to study the notions of randomness beyond Martin-Löf randomness. These are the relativised versions "Kurtz random relative to K", "Schnorr random relative to K", and "Martin-Löf random relative to K" where K is the halting problem or any other creative set. In addition, the three independently defined notions of "Demuth random", "weakly Demuth random" and "strongly random" are considered. Strong randomness is by some authors considered to be the next counterpart of Kurtz randomness, although it is not the relativised version; therefore they call Kurtz random also "weakly random" and strongly random also "weakly 2-random" (see, for example, Nies [18]). Strong randomness [12,22] has various nice characterisations, in particular the following: A is strongly random iff A is Martin-Löf random and forms a minimal pair with K with respect to Turing reducibility [6, Footnote 2]. For these notions, in order to quantify the degree of non-randomness of a sequence, one studies from which value $f(m)$ onwards all initial segments can be compressed by m bits. That is, one looks at functions f such that $C(A(0)A(1)\dots A(n)) \leqslant n - m$ for all $n > f(m)$; here f might also be an upper bound of the least possible point with this property as one might want to have that f is in a certain Turing degree.

Looking for this type of characterisation for randomness notions between Martin-Löf randomness and 2-randomness (that is, Martin-Löf randomness relative to K) is quite natural. This is because we know [15,19] that 2-randomness coincides with "Kolmogorov randomness", a notion that is defined by the absence of any f as above.

The results are summarised in the following list:

- A is not 2-random iff there is $f \leqslant_T A \oplus K$ such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n-m$;
- A is not strongly random iff there are $f \leqslant_T A$ and $g \leqslant_T K$ such that for all m and all $n > f(g(m))$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n - m$;
- A is not Kurtz random relative to K iff there is $f \leqslant_T K$ such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n - m$;
- A is not Schnorr random relative to K iff there is $f \leqslant_T K$ such that for infinitely many m and all $n > f(m)$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n - m$;
- A is not Demuth random iff there is an ω-r.e. function f such that for infinitely many m and all $n > f(m)$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n - m$;
- A is not weakly Demuth random iff there are $f \leqslant_T A$ and an ω-r.e. function g such that for all m and all $n > f(g(m))$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n - m$;
- $A \geqslant_T K$ or A is not Martin-Löf random iff there is an A-recursive function f such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n - m$;
- A is not strongly Kurtz random iff there is a recursive function f such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\dots A(n)) \leqslant n - m$.

So the main differences between these characterisations are how the bound function is formed and whether the compressibility condition holds for infinitely many or for all m. Note that due to finite modifications of f it would be equivalent to postulate the condition for all m or for almost all m. Several proofs make use of this fact.

Although the unrelativised versions of Kurtz randomness and Schnorr randomness do not admit such a characterisation in terms of plain Kolmogorov complexity, Bienvenu and Merkle [1] gave one in terms of Kolmogorov complexity defined by computable machines. There is a close connection between the plain Kolmogorov complexity C and prefix-free Kolmogorov complexity H, which will be formalised in Remark 2. This connection helps to establish many bounds obtained for C also for H.

For the scientific background of this paper, the reader is referred to the usual textbooks on recursion theory [20,21,25] and algorithmic randomness [2,5,13,18].

2. Notation and basic definitions

For the sake of completeness, we repeat the definition of Kolmogorov complexity.

Definition 1. Let M be a Turing machine mapping binary strings to binary strings. Then the (*plain*) *Kolmogorov complexity with respect to M* is the function C_M from strings to integers defined by

$$C_M(\tau) = \min\{|\sigma|:\ M(\sigma) = \tau\}.$$

By effective codings of Turing machines there exists a machine V which is *optimal* for C, i.e. $C_V \leqslant C_M + O(1)$ for each machine M. In the following we let C denote C_V for a fixed optimal machine. Similarly, there exists a machine U with