Information and Computation 234 (2014) 68–106

Contents lists available at ScienceDirect

Information and Computation

www.elsevier.com/locate/yinco

Least upper bounds for probability measures and their applications to abstractions

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ARTICLE INFO

Article history:
Received 28 November 2011
Received in revised form 22 May 2013
Available online 18 December 2013

ABSTRACT

Least upper bounds play an important role in defining the semantics of programming languages, and in abstract interpretations. In this paper, we identify conditions on countable ordered measurable spaces that ensure the existence of least upper bounds for all sets of probability measures. These conditions are shown to be necessary as well — for any measurable space not satisfying these conditions, there are (finite) sets of probability measures for which no least upper bound exists. For measurable spaces meeting these conditions, the existence of least upper bounds is established constructively. Based on this least upper bound construction, we present a novel abstraction method applicable to Discrete Time Markov Chains (DTMCs), Markov Decision Processes (MDPs), and Continuous Time Markov Chains (CTMCs). The main advantage of the new abstraction techniques is that the resulting abstract models are *purely probabilistic* that may be more amenable to automated analysis than models with both nondeterministic and probabilistic transitions which arise from previously known abstraction techniques.

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1. Introduction

Order structures have served a pivotal role in program semantics and analysis of probabilistic systems. Starting from the work of Saheb-Djahromi [25], and further developed by Jones [13], an ordering relation on probability measures has been used in developing a denotational semantics for probabilistic programs. This ordering also plays a central role in defining the notion of simulation for probabilistic systems. In a probabilistic model, a transition specifies a probability measure on its successor states. One transition then simulates another if the probability measures they specify are related by the ordering on measures. In this manner, simulation and bisimulation relations were first defined for Deterministic Time Markov Chains (DTMC), and Markov Decision Processes (MDP) [14], and subsequently extended to Continuous Time Markov Chains (CTMC) [3]. Therefore, in all these settings, a set of transitions is abstracted by a transition if it is an upper bound for the probability measures specified by the set of transitions being abstracted.

Based on this observation that an upper bound probability measure can serve as a correct abstraction, we investigate the construction of least upper bounds for sets of probability measures and their application to abstraction. We begin by observing that in general, measures (even over simple finite spaces) do not have least upper bounds. We therefore look for a class of measurable spaces for which the existence of least upper bounds is guaranteed for arbitrary sets of measures. Since the ordering relation on measures is induced from the underlying partial order on the space over which the measures

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0890-5401/$ – see front matter © 2014 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.ic.2013.12.003

are considered, our goal is to identify conditions on the underlying partial order that are sufficient to prove the existence of least upper bounds.

For general measurable spaces, the order structure on measures is very sensitive to the specific collection of measurable subsets (σ-algebra). This makes a characterization of the necessary and sufficient conditions (for the existence of least upper bounds) for any measurable space particularly challenging. We overcome this problem by developing a new representation theorem that shows how any countable measurable space can be reduced to a countable discrete measurable space.[2] More precisely, for any countable measurable space $\mathcal{X}$ (over a set with an underlying preorder), we show how to construct a countable set $\mathsf{At}(\mathcal{X})$ and a partial order over it such that there is an *order-preserving isomorphism* between the (ordered) spaces of probability measures over $\mathcal{X}$ and the discrete probability measures over $\mathsf{At}(\mathcal{X})$. The construction of the set $\mathsf{At}(\mathcal{X})$ is based both on the measurable subsets of $\mathcal{X}$ as well as its underlying preorder. In light of the intricate sensitivity of the order structure of probability measures to the underlying order and measurable subsets, this representation theorem provides a powerful simplifying tool.

Given the representation theorem, we can, without loss of generality, restrict our attention to countable discrete measurable spaces with an underlying partial order. We define conditions on the underlying partial order that intuitively correspond to requiring the Hasse diagram of the partial order to have a "tree-like" structure. We establish these conditions to be necessary — the underlying partial order on any measurable space admitting least upper bounds is proven to satisfy the requirements of being "tree-like". For measurable spaces meeting these conditions, we present a construction of least upper bounds for arbitrary collections of probability measures.

Next, we present a new method to abstract probabilistic transition systems based on our least upper bound construction. Abstractions are constructed on the basis of an equivalence relation on the set of (concrete) states of the system being abstracted with the equivalence classes forming the (abstract) states of the abstract model (thus, the abstract model collapses all equivalent states into one). In previously constructed abstractions, each abstract state has multiple (nondeterministic) transitions corresponding to the transitions of each of the concrete states in the equivalence class. However, in our construction, the transition out of an abstract state is taken to be the *least upper bound measure* of the transitions from each of the concrete states it "abstracts." This yields a single outgoing transition resulting in an abstract model that is *purely probabilistic* which does not have any nondeterminism. These abstraction constructions are presented and proved correct for DTMCs, MDPs and CTMCs.

A few salient features of our abstract models bear highlighting. First, the fact that least upper bounds are used in the construction implies that for a particular equivalence relation on concrete states and partial order on the abstract states, the abstract model constructed is finer than (*i.e.*, can be simulated by) any purely probabilistic model that can serve as an abstraction. Thus, for verification purposes, our model is the most precise purely probabilistic abstraction on a chosen state space. Second, the set of abstract states is not completely determined by the equivalence classes of the relation on concrete states; there is freedom in the choice of states that are above the equivalence classes in the partial order. However, for any such choice that respects the "tree-like" requirement on the underlying partial order, the resulting model will be exponentially smaller than the existing proposals of [9,16]. We show that there are instances where we can get more precise results than the abstraction schemes of [9,16] while using significantly fewer states (see Example 6.9 and Section 7.2). Third, the abstract models we construct are purely probabilistic which makes model checking easier. Additionally, these abstractions can potentially be verified using statistical techniques which do not work when there is nondeterminism [30,29,27]. Finally, CTMC models with nondeterminism, called CTMDP, are known to be difficult to analyze [2]. Specifically, the measure of time-bounded reachability can only be computed approximately through an iterative process; therefore, there is only an approximate algorithm for model-checking CTMDPs against CSL. On the other hand, there is a theoretically exact solution to the corresponding model-checking problem for CTMCs by reduction to the first order theory of reals [1].

Related work. Abstractions have been extensively studied in the context of probabilistic systems. General issues and definitions of good abstractions are presented in [14,11,12,22]. Specific proposals for families of abstract models include Markov Decision Processes [14,6,7], systems with interval ranges for transition probabilities [14,22,9,16], abstract interpretations [21], 2-player stochastic games [17], and expectation transformers [19]. Recently, theorem-prover based algorithms for constructing abstractions of probabilistic systems based on predicates have been presented [28]. All the above proposals construct models that exhibit both nondeterministic and probabilistic behavior. The abstraction method presented in this paper constructs purely probabilistic models.

The order structure on measures was first studied extensively in the context of the probabilistic powerdomain construction [25,13] for denotational semantics. This theory is based on a particular σ-algebra (the Borel field induced by Scott-open sets) and is concerned with the existence of least upper bounds only for increasing chains (directed complete partial orders); on the other hand, the work presented here considers arbitrary σ-algebras and least upper bounds of all subsets. Stronger order structures (subcategories of DCPO) have to be considered for the probabilistic powerdomain construction to satisfy the properties of a commutative monad and a significant challenge encountered in this development has been the difficulty in preserving any lattice-like structure on measures (*cf.* [15]). While none of the order structures studied are as strong as a join semi-lattice that is characterized in this paper, an interesting point of comparison is that the finite tree-reversal posets

[2] A discrete measurable space is one where every subset of the universe is measurable.

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