



Curves that must be retraced

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ABSTRACT

We exhibit a polynomial time computable plane curve Γ that has finite length, does not intersect itself, and is smooth except at one endpoint, but has the following property. For every computable parametrization f of Γ and every positive integer m , there is some positive-length subcurve of Γ that f retraces at least m times. In contrast, every computable curve of finite length that does not intersect itself has a constant-speed (hence non-retracing) parametrization that is computable relative to the halting problem.

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1. Introduction

A curve is a mathematical model of the path of a particle undergoing continuous motion. Specifically, in a Euclidean space $\mathbb{R}^n$, a curve is the range Γ of a continuous function $f : [a, b] \rightarrow \mathbb{R}^n$ for some $a < b$. The function f , called a *parametrization* of Γ , clearly contains more information than the pointset Γ , namely, the precise manner in which the particle “traces” the points $f(t) \in \Gamma$ as t , which is often considered a time parameter, varies from a to b . When the particle’s motion is algorithmically governed, the parametrization must be computable (as a function on the reals; see below).

This paper shows that the geometry of a curve Γ may force every *computable* parametrization f of Γ to retrace various parts of its path (i.e., “go back and forth along Γ ”) many times, even when Γ is an efficiently computable, smooth, finite-length curve that does not intersect itself. In fact, our main theorem exhibits a plane curve $\Gamma \subseteq \mathbb{R}^2$ with the following properties.

1. Γ is *simple*, i.e., it does not intersect itself.
2. Γ is *rectifiable*, i.e., it has finite length.
3. Γ is *smooth except at one endpoint*, i.e., Γ has a tangent at every interior point and a 1-sided tangent at one endpoint, and these tangents vary continuously along Γ .
4. Γ is *polynomial time computable* in the strong sense that there is a polynomial time computable position function $\vec{s} : [0, 1] \rightarrow \mathbb{R}^2$ such that the velocity function $\vec{v} = \vec{s}'$ and the acceleration function $\vec{a} = \vec{v}'$ are polynomial time computable; the total distance traversed by $\vec{s}$ is finite; and $\vec{s}$ parametrizes Γ , i.e., $\text{range}(\vec{s}) = \Gamma$.

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5. Γ must be retraced in the sense that every parametrization $f : [a, b] \rightarrow \mathbb{R}^2$ of Γ that is computable in *any* amount of time has the following property. For every positive integer m , there exist disjoint, closed subintervals $I_0, \dots, I_m$ of $[a, b]$ such that the curve $\Gamma_0 = f(I_0)$ has positive length and $f(I_i) = \Gamma_0$ for all $1 \leq i \leq m$. (Hence f retraces Γ_0 at least m times.)

The terms “computable” and “polynomial time computable” in properties 4 and 5 above refer to the “bit-computability” model of computation on reals formulated in the 1950s by Grzegorzczuk [9] and Lacombe [17], extended to feasible computability in the 1980s by Ko and Friedman [13] and Kreitz and Weihrauch [16], and expositied in the recent paper by Braverman and Cook [4] and the monographs [5, 14, 20, 23]. As will be shown here, condition 4 also implies that the pointset Γ is polynomial time computable in the sense of Brattka and Weihrauch [2]. (See also [3, 4, 23].)

A fundamental and useful theorem of classical analysis states that every simple, rectifiable curve Γ has a *normalized constant-speed parametrization*, which is a one-to-one parametrization $f : [0, 1] \rightarrow \mathbb{R}^n$ of Γ with the property that $f([0, t])$ has arclength tL for all $0 \leq t \leq 1$, where L is the length of Γ . (A simple, rectifiable curve Γ has exactly two such parametrizations, one in each direction, and standard terminology calls either of these *the* normalized constant-speed parametrization $f : [0, 1] \rightarrow \mathbb{R}^n$ of Γ . The constant-speed parametrization is also called the *parametrization by arclength* when it is reformulated as a function $f : [0, L] \rightarrow \mathbb{R}^n$ that moves with constant speed 1 along Γ .) Since the constant-speed parametrization does not retrace any part of the curve, our main theorem implies that this classical theorem is not entirely constructive. Even when a simple, rectifiable curve has an efficiently computable parametrization, the constant-speed parametrization need not be computable.

In addition to our main theorem, we prove that every simple, rectifiable curve Γ in $\mathbb{R}^n$ with a computable parametrization has the following two properties:

- I. The length of Γ is lower semicomputable.
- II. The constant-speed parametrization of Γ is computable relative to the length of Γ .

These two things are not hard to prove if the computable parametrization is one-to-one, (in fact, they follow from results of Müller and Zhao [19] in this case) but our results hold even when the computable parametrization retraces portions of the curve many times.

Taken together, I and II have the following two consequences.

1. The curve Γ of our main theorem has a finite length that is lower semi-computable but not computable. (The existence of polynomial-time computable curves with this property was first proven by Ko [15].)
2. Every simple, rectifiable curve Γ in $\mathbb{R}^n$ with a computable parametrization has a constant-speed parametrization that is Δ_2^0 -computable, i.e., computable relative to the halting problem. Hence, the existence of a constant-speed parametrization, while not entirely constructive, is constructive relative to the halting problem.

2. Length, computability, and complexity of curves

In this section, we summarize basic terminology and facts about curves. As we use the terms here, a *curve* is the range Γ of a continuous function $f : [a, b] \rightarrow \mathbb{R}^n$ for some $a < b$. The function f is called a *parametrization* of Γ . Each curve clearly has infinitely many parametrizations.

A curve is *simple* if it has a parametrization that is one-to-one, i.e., the curve “does not intersect itself”. The length of a simple curve Γ is defined as follows. Let $f : [a, b] \xrightarrow{1-1} \mathbb{R}^n$ be a one-to-one parametrization of Γ . For each *dissection* $\vec{t}$ of $[a, b]$, i.e., each tuple $\vec{t} = (t_0, \dots, t_m)$ with $a = t_0 < t_1 < \dots < t_m = b$, define the *f-t-approximate length* of Γ to be

$$\mathcal{L}_{\vec{t}}^f(\Gamma) = \sum_{i=0}^{m-1} |f(t_{i+1}) - f(t_i)|.$$

Then the *length* of Γ is

$$\mathcal{L}(\Gamma) = \sup_{\vec{t}} \mathcal{L}_{\vec{t}}^f(\Gamma),$$

where the supremum is taken over all dissections $\vec{t}$ of $[a, b]$. It is easy to show that $\mathcal{L}(\Gamma)$ does not depend on the choice of the one-to-one parametrization f , i.e., that the length is an intrinsic property of the pointset Γ .

In Sections 4 and 5 of this paper, we use a more general notion of length, namely, the one-dimensional Hausdorff measure $\mathcal{H}^1(\Gamma)$, which is defined for every set $\Gamma \subseteq \mathbb{R}^n$. We refer the reader to [7] for the definition of $\mathcal{H}^1(\Gamma)$. It is well known that $\mathcal{H}^1(\Gamma) = \mathcal{L}(\Gamma)$ holds for every simple curve Γ .

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