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The classification of abelian groups generated by time-varying automata and by Mealy automata over the binary alphabet

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ABSTRACT

For every natural number n, we classify abelian groups generated by an n-state time-varying automaton over the binary alphabet, as well as by an n-state Mealy automaton over the binary alphabet.

1. Introduction

In the theory of computation time-varying automata over a finite alphabet are finite-state transducers which constitute a natural generalization of Mealy-type automata as they allow to change both the transition function and the output function in successive steps of processing input sequences of letters into output sequences (see [10]). These automata in turn constitute a subclass in the class of time-varying automata over a changing alphabet (see [14]). In group theory all these types of transducers turned out to be a useful tool for defining and studying groups of automorphisms of certain rooted trees, which helped in the discovery of interesting geometric and algebraic properties and the dynamics of various types of groups associated with their actions on these trees. The intensive study of Mealy automata in relation to automorphism groups has continued for last three decades, and the extraordinary properties of these groups are discussed in a great deal of papers (for the comprehensive works see [6–9,11]). On the other hand, the investigation of automorphism groups as groups defined by time-varying automata is a quite new approach, which we introduced in [14] (for other results on this subject see [3,4,15,16,18,19]).

If an automaton A is invertible, then each of its states defines an automorphism of the corresponding rooted tree. The group generated by these automorphisms (i.e. by automorphisms corresponding to all the states of A) is called the group generated by the automaton A and is usually denoted by $G(A)$.

Definition 1. Let n and k be natural numbers and let G be an abstract group. We say that G is generated by an n-state time-varying (Mealy) automaton over a k-letter alphabet if there is an invertible time-varying (Mealy) automaton A with an n-element set of states which works over a k-letter alphabet such that G is isomorphic to the group $G(A)$ generated by this automaton.

Given two natural numbers n, k and a class of abstract groups, the natural question arises: which groups from this class are generated by an n-state (time-varying or Mealy) automaton over a k-letter alphabet? This problem was studied so far only in the case of Mealy-type automata and it turned out to be difficult even for small values of n and k and for classes

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containing algebraically well known constructions, such as abelian groups, free groups, free products of finite groups, and many others.

It is worth noting that there is no general method for deducing even the basic algebraic properties of the group generated by an automaton directly from the structure of this automaton. For example, for a long time it was an open problem whether a free non-abelian group is generated by a Mealy automaton (see [5] solving this problem). However, a candidate for the solution was known and studied since 80s of the last century (the so-called Aleshin–Vorobets automaton—see [1,13]). Until now, we do not know if the free non-abelian group of rank two is generated by a 2-state Mealy automaton or by a 2-state time-varying automaton over a finite alphabet. On the other hand, we provided for this group a natural construction of a 2-state time-varying automaton over an unbounded changing alphabet (see [16]).

The full classification of groups generated by Mealy automata is known only in the simplest non-trivial case, i.e. when $n = k = 2$ ([7]). Apart from that, for the classes of abelian groups and finite groups the solutions for Mealy automata are known only in the case $(n, k) = (3, 2)$ (see Theorems 3, 4 in the extensive work [2] devoted to the classification of all groups generated by 3-state Mealy automata over the binary alphabet).

2. The results

The goal of this paper is providing for every natural number n the full classification of abelian groups generated by an n-state time-varying automaton over the binary alphabet as well as by an n-state Mealy automaton over the binary alphabet.

Let us denote the following classes of abelian groups:

- $\mathcal{TVA}(n)$—the class of abelian groups generated by an n-state time-varying automaton over the binary alphabet,
- $\mathcal{MA}(n)$—the class of abelian groups generated by an n-state Mealy automaton over the binary alphabet,
- $\mathcal{AB}_2(n)$—the class of abelian groups of rank not greater than n in which the torsion part is a 2-group,
- $\mathcal{FA}(n)$—the class of free abelian groups of rank not greater than n,
- $\mathcal{EA}_2(n)$—the class of elementary abelian 2-groups of rank not greater than n.

The main result of the paper is the following characterization.

Theorem 1. $\mathcal{TVA}(1) = \mathcal{EA}_2(1)$, *and if* $n > 1$, *then* $\mathcal{TVA}(n) = \mathcal{AB}_2(n)$.

Theorem 2. $\mathcal{MA}(n) = \mathcal{FA}(n-1) \cup \mathcal{EA}_2(n)$. *In particular, there are exactly* $2n$ *abelian groups generated by an* n*-state Mealy automaton over the binary alphabet.*

For the proof of Theorem 1 (Section 5), we find at first some restrictions on abelian groups generated by time-varying automata over the binary alphabet. These restrictions imply the inclusion $\mathcal{TVA}(n) \subseteq \mathcal{AB}_2(n)$ for each $n \geq 1$, as well as the equality $\mathcal{TVA}(1) = \mathcal{EA}_2(1)$. Next, for every $n \geq 1$ and every group $G \in \mathcal{AB}_2(n)$ of rank n, we provide an explicit construction of a time-varying automaton A over the binary alphabet which generates G (see Propositions 5–7). In the construction of the automaton A, we distinguish between three cases: where the group G is cyclic, where G is non-cyclic and has a non-trivial torsion part, and finally, where G is a non-cyclic, torsion-free group. Note that in the last case G must be a free abelian group, which follows from the property that every finitely generated, abelian group is a direct sum of cyclic groups. To derive the required property of our construction, we use the language of wreath recursions, which is popular in the study of groups generated by automata, and which arises from an embedding of the group $G(A)$ into the permutational wreath product (for more on the method involving wreath recursions in the case of Mealy automata see, for example, [7,11], and in the case of time-varying automata—see [19]).

In the proof of Theorem 2 (Section 6), for an arbitrary group G generated by a Mealy automaton over the binary alphabet, we deduce certain relations between the first level stabilizer of G, the set I_G of involutions and the set G^2 of squares. We show (Proposition 8) a quite unexpected dichotomy, which holds in the case when G is abelian. Namely, the relations imply in this case that G is either a free abelian group or an elementary abelian 2-group. Next, we introduce and study a construction of the Mealy automata generating elementary abelian 2-groups. We also use the known construction of the Mealy automata generating free abelian groups (so-called "sausage" automata—see [7]), as well as the restriction for the rank of free abelian groups generated by Mealy automata over the binary alphabet (see Proposition 3.1 from [17]).

We hope that the above characterizations, together with the involved constructions and the methods presented in the proofs, will encourage to develop further study of the groups generated by time-varying automata. This may be useful when investigating some important, computational problems, which are open in the class of groups generated by time-varying automata, but have a simple solution in the class of groups generated by Mealy automata. Let us mention the word problem for groups (which is decidable in the latter class). It is known that there is a time-varying automaton A over an arbitrary, unbounded changing alphabet such that the group $G(A)$ has undecidable word problem. This follows from the fact that there exist finitely generated, residually finite groups with undecidable word problem, as well as from some unconstructive method of defining an automaton realization for an arbitrary, finitely generated, residually finite group (the so-called diagonal realization—see [14,18]). However, there is not known any explicit and naturally defined construction of an automaton