Information Processing Letters 116 (2016) 660–667

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

On the algorithmic complexity of zero-sum edge-coloring

Ali Dehghan [a], Mohammad-Reza Sadeghi [b,*]

[a] Systems and Computer Engineering Department, Carleton University, Ottawa, Canada
[b] Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran

ARTICLE INFO

Article history:
Received 7 July 2015
Received in revised form 21 May 2016
Accepted 24 June 2016
Available online 29 June 2016
Communicated by Ł. Kowalik

Keywords:
Zero-sum edge-coloring
Zero-sum flow
Zero-sum vertex flow
Computational complexity

ABSTRACT

A zero-sum k-flow for a graph G is a vector in the null space of the 0,1-incidence matrix of G such that its entries belong to $\{\pm 1, \cdots, \pm(k-1)\}$. Also, a *zero-sum vertex k-flow* is a vector in the null space of the 0,1-adjacency matrix of G such that its entries belong to $\{\pm 1, \cdots, \pm(k-1)\}$. Furthermore, a *zero-sum k-edge-coloring* of a simple graph G is a vector in the null space of the 0,1-incidence matrix of G such that its entries belong to $\{\pm 1, \cdots, \pm(k-1)\}$ and this vector is a proper edge coloring (adjacent edges receive distinct colors) for G. In this work, we show that there is a polynomial time algorithm to determine whether a given graph G has a zero-sum edge-coloring. Also, we prove that there is no constant bound k, such that for a given bipartite graph G, if G has a zero-sum vertex flow, then G has a zero-sum vertex k-flow. Furthermore, we show that for a given bipartite $(2,3)$-graph G, it is **NP**-complete to determine whether G has a zero-sum vertex 3-flow.

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1. Introduction

Let G be a simple graph, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Also, for every $v \in V(G)$, $d_G(v)$ and $N(v)$ denote the degree of v and the neighbor set of v, respectively. We denote the maximum degree of G by $\Delta(G)$. The *adjacency matrix* of a simple graph G is an $n \times n$ matrix $A = [a_{ij}]$ where n is the number of vertices, such that $a_{ij} = 1$ if the vertex v_i is adjacent with the vertex v_j and 0 otherwise. The *incidence matrix* of a simple graph G is an $n \times m$ matrix $B = [b_{ij}]$ such that $b_{ij} = 1$ if the vertex v_i and edge e_j are incident and 0 otherwise. The *null space* of a matrix A is the set of all vectors x for which $Ax = 0$. We follow [23] for terminology and notations which are not defined here.

Investigating the structures of the kernel of the incidence matrix and the kernel of the adjacency matrix of any undirected graph is an active area in linear algebra, for example, Villarreal proved that the null space of the incidence matrix of every graph has a basis whose elements have entries in $\{-2, -1, 0, 1, 2\}$ [20]. For more examples, see [4,6,8,11,15,20]. A *zero-sum k-flow* for a graph G is a vector in the null space of the 0,1-incidence matrix of G such that its entries belong to $\{\pm 1, \cdots, \pm(k-1)\}$. Note that zero-sum k-flow also, is called zero-sum edge k-flow. Determining whether a given graph G has a zero-sum flow is in **P** [2]. Also, a graph G has zero-sum 2-flow if and only if G is Eulerian with even size (even number of edges) in each component, hence deciding whether a given graph G has zero-sum 2-flow is in **P** [22]. A graph G is a $(d, d+s)$-graph if the degree of every vertex of G lies in the interval $[d, d+s]$. It was shown that it is **NP**-complete to determine whether a given $(3,4)$-graph has a zero-sum 3-flows (has entries in $\{-2, -1, 1, 2\}$) [5]. Also, it is proved that it was **NP**-complete to determine whether a given $(2,3)$-graph has a zero-sum 4-flows [5].

* Corresponding author.
E-mail addresses: alidehghan@sce.carleton.ca (A. Dehghan), msadeghi@aut.ac.ir (M.-R. Sadeghi).

http://dx.doi.org/10.1016/j.ipl.2016.06.010
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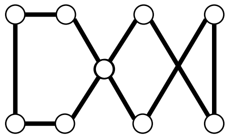

Fig. 1. An example of graph with some zero-sum flows and no zero-sum edge-coloring.

1.1. Zero-sum k-edge-coloring

A *magic square* is an arrangement of distinct integers in a square grid, where the numbers in each row, and in each column, and the numbers in the forward and backward main diagonals, all add up to the same number and the *semi-magic squares* is an arrangement of distinct integers in a square grid, when all row and column sums are constrained to have the same magic sum [13]. Motivated by the notion of semi-magic squares, semi-magic edge labelings were introduced by Stewart [19]. Stewart calls a connected graph semi-magic if there is a labeling of the edges with integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v [19]. For more information we refer the reader to a survey on graph labeling by Gallian [7]. On the other hand, there are different types of edge colorings for graphs, see [17,18]. Motivated by the notion of semi-magic squares, we consider a new variant of edge coloring which is zero-sum. A *zero-sum k-edge-coloring* of a simple graph G is a vector in the null space of the 0,1-incidence matrix of G such that its entries belong to $\{\pm 1, \cdots, \pm(k-1)\}$ and this vector is a proper edge coloring for G. There are graphs with some zero-sum flows such that they do not have any zero-sum edge-coloring (for example see Fig. 1). In the following theorem we characterized all graphs with zero-sum edge-coloring. In fact, for a given graph G we give a necessary and sufficient condition, checkable in polynomial time, to have any zero-sum edge-coloring.

Theorem 1. *Determining whether a given graph G has a zero-sum edge-coloring is in* **P**.

Let G be a given connected graph with at least two vertices, it has a zero-sum 2-edge-coloring if and only if it is an even cycle. By Vizing's theorem [21], the chromatic index of a graph G is equal to either $\Delta(G)$ or $\Delta(G)+1$. It was shown that it is **NP**-hard to determine the chromatic index of a cubic graph [10]. Let G be a connected 3-regular graph and has a zero-sum 4-edge-coloring. Since the sum of the labels of all three edges incident with a vertex v, $v \in V(G)$ is zero and these labels are different, so the labels of these edges are $\{1, 2, -3\}$ or $\{-1, -2, 3\}$. Without loss of generality suppose that the labels of the three edges incident with a vertex $v \in V(G)$ is $\{1, 2, -3\}$. Since G is connected, therefore for every vertex $u \in V(G)$ the labels of the three edges incident with a vertex u is $\{1, 2, -3\}$. Consequently, G has a zero-sum 4-edge-coloring if and only if the chromatic index of G is 3. Therefore, it is **NP**-complete to decide whether a 3-regular graph G has a zero-sum 4-edge-coloring.

Theorem 2. *For a given 3-regular graph G, it is* **NP***-complete to determine whether G has a zero-sum 4-edge-coloring.*

Remark 1. It was shown that it is **NP**-hard to determine the chromatic index of an r-regular graph for any $r \geq 3$ [12]. Therefore, for every $r \geq 2$, it is easy to see that for a given $(2r)$-regular graph G it is **NP**-complete to decide whether G has a zero-sum $(r+1)$-edge-coloring.

Remark 2. Note that for a given graph G, a zero-sum edge-coloring of G is an assignment of non-zero integers to the edges of G such that the sum of the labels of all edges incident with each vertex is zero and adjacent edges receive distinct labels. In fact, in zero-sum edge-coloring we are searching among the vectors in the null space of the incidence matrix of a graph, to find a vector which is a proper edge coloring for G.

1.2. Zero-sum vertex flow

A *zero-sum vertex flow* for a graph G, is an integral nowhere-zero element in the null space of the 0,1-adjacency matrix of G. A *zero-sum vertex k-flow* is a vector in the null space of the 0,1-adjacency matrix of G such that its entries belong to $\{\pm 1, \cdots, \pm(k-1)\}$.

Finding an integral element in the null space of the 0,1-adjacency matrix of a graph is an important topic and has many applications in linear algebra, see [3,9]. Also, there are interesting connections between zero-sum k-flow and zero-sum vertex k-flow. For instance, for a given bipartite graph G, consider the bipartite graph $G^{\frac{1}{2}}$ ($G^{\frac{1}{2}}$ is obtained from G by replacing each edge with a path with exactly one inner vertex). On can see that $G^{\frac{1}{2}}$ has a zero-sum vertex k-flow if and only if G has a zero-sum k-flow. Motivated by the importance of the finding an integral element in the null space of a matrix, one can ask the following about the adjacency matrix of graphs.

Problem 1. Is there a polynomial time algorithm to decide whether a given graph G has a zero-sum vertex flow?

We have the following about the zero-sum vertex flow in bipartite graphs.

Theorem 3. *There is no constant bound k, such that for a given* $(2, 3)$*-bipartite graph G, if G has a zero-sum vertex flow, then G has a zero-sum vertex k-flow.*

If G is a graph with zero-sum vertex 2-flow, then the degree of every vertex is even. It is easy to determine whether a given 2-regular graph G has a zero-sum vertex 2-flow. Next, we consider the computational complexity of determining whether a given bipartite graph has a zero-sum vertex 3-flow or 2-flow.

Theorem 4. *We have the following:*

(i) *For a given bipartite* $(2, 3)$*-graph G, it is* **NP***-complete to determine whether G has a zero-sum vertex 3-flow.*
(ii) *For a given bipartite* $(2, 4)$*-graph G, it is* **NP***-complete to determine whether G has a zero-sum vertex 2-flow.*

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