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The metric dimension for resolving several objects

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ABSTRACT

A set of vertices S is a *resolving set* in a graph if each vertex has a unique array of distances to the vertices of S. The natural problem of finding the smallest cardinality of a resolving set in a graph has been widely studied over the years. In this paper, we wish to resolve a set of vertices (up to ℓ vertices) instead of just one vertex with the aid of the array of distances. The smallest cardinality of a set S resolving at most ℓ vertices is called ℓ-set-metric dimension. We study the problem of the ℓ-set-metric dimension in two infinite classes of graphs, namely, the two dimensional grid graphs and the n-dimensional binary hypercubes.

1. Introduction

In this paper, a graph G is finite, undirected, simple and connected. As usual, we denote its vertex set by V and the set of edges by E. The distance between two vertices $u, v \in V$ (that is, the number of edges in any shortest path joining u and v) is denoted by $d(u, v) = d_G(u, v)$. Let $N(v) = \{u \in V \mid d(u, v) = 1\}$ for $v \in V$. The *Cartesian product* of graphs $G = (V, E)$ and $H = (V', E')$, denoted by $G \square H$, is the graph with vertex set $V \times V' = \{(a, b) \mid a \in V, b \in V'\}$, where (a, b) is adjacent to (u, v) if $a = u$ and the edge $\{b, v\} \in E'$, or $b = v$ and $\{a, u\} \in E$. The distance $d((a, b), (u, v)) = d_G(a, u) + d_H(b, v)$.

Let $S \subseteq V$ and denote its cardinality by $|S|$. Let us write S as an ordered set $S = (s_1, s_2, \ldots, s_{|S|})$. For any $x \in V$, we denote by

$$\mathcal{D}(x) = \mathcal{D}_S(x) = (d(x, s_1), d(x, s_2), \ldots, d(x, s_{|S|}))$$

the distance array of x with respect to S. If $\mathcal{D}_S(x) \neq \mathcal{D}_S(y)$ for any two distinct vertices x and y in V, then S is called a *resolving set*. The concept of a resolving set was introduced independently by Slater [15] and Harary and Melter [8]. Resolving sets are widely studied [5,3,4,9,1,6,13] and these sets have many connections to other diverse problems, see for example, network discovery and verification [2], robot navigation [10] and connected joins in graphs [14]. In [15], each $s_i \in S$ is considered as a site for a sonar station, and the location of an object (like an intruder in $x \in V$) is then uniquely determined using its distances to stations in $\mathcal{D}(x)$.

In this paper, we consider the situation where there can be several objects whose locations (the set $X \subseteq V$) we want to determine simultaneously. Naturally, here each sonar $s_i \in S$ measures the distance to the *closest* vertex in the object set $X \subseteq V$ (there can be several objects at that particular distance), but reveals no further information on the locations or the cardinality of X. Finding several objects has earlier been considered in other contexts of sensor networks, like in the case of identifying codes and locating–dominating sets, where the sensors can detect objects within a fixed radius, see [7,12] and also the list in [11].

For any $X \subseteq V$ and $v \in V$, denote $d(v, X) = \min\{d(v, x) \mid x \in X\}$. Furthermore, for any $X \subseteq V$, let the distance array

$$\mathcal{D}(X) = \mathcal{D}_S(X) = (d(s_1, X), d(s_2, X), \ldots, d(s_{|S|}, X)).$$

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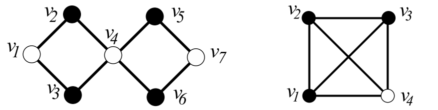

(a) The graph G_7 (b) The graph $\mathcal{K}_4$

Fig. 1. The set S consists of the black vertices.

We write in short, $\mathcal{D}(\{x_1,\dots,x_k\})=\mathcal{D}(x_1,\dots,x_k)$. Hence $\mathcal{D}(x)$ means the same distance array as before.

Definition 1. Let $G=(V,E)$ be a finite, undirected, simple and connected graph. Let further ℓ be an integer such that $1\le \ell\le |V|$. A subset $S\subseteq V$ is called an *ℓ-resolving set* (or an *ℓ-set resolving set*) if

$$\mathcal{D}(X)\neq\mathcal{D}(Y)$$

for any two distinct and nonempty subsets $X,Y\subseteq V$ with $|X|\le\ell$ and $|Y|\le\ell$.

The minimum cardinality of an ℓ-resolving set of G is called the *ℓ-set-metric dimension* of G and it is denoted by $\beta_\ell(G)$. An ℓ-resolving set of cardinality $\beta_\ell(G)$ is called an *ℓ-set-metric basis* of G. Clearly, a 1-resolving set is the usual resolving set, and the set $S=V$ is always an ℓ-resolving set for all $1\le\ell\le|V|$.

Example 2.

(i) Let us consider the graph of Fig. 1(a). Take the set $S=\{v_2,v_3,v_5,v_6\}$. It is easy to check that S is a 1-resolving set, and $\mathcal{D}(v_4)=(1,1,1,1)$. If we receive the distance array $(1,1,1,1)$, we immediately conclude that the object (like an intruder) is in v_4. However, if there are *two* objects (intruders), say in v_1 and v_7, we can falsely make that decision and no intruder is found, since also $\mathcal{D}(v_1,v_7)=(1,1,1,1)$.

(ii) Denote a path on $n\ge 2$ vertices by P_n and write the vertices as an ordered set $P_n=(v_1,v_2,\dots,v_n)$. The set $S=\{v_1,v_n\}$ is a 2-resolving set as we will show next. Let $X\subseteq\{v_1,\dots,v_n\}$ and $1\le|X|\le 2$. Now $\mathcal{D}(X)=(a,b)$ for some $0\le a,b\le n-1$ (here $S=(v_1,v_n)$ is considered as an ordered set). If $a+b=n-1$, then there X consists of one vertex, namely, v_{1+a}. On the other hand, if $a+b<n-1$, then there are two vertices in X, namely, $X=\{v_{i+a},v_{n-1-b}\}$. Consequently, $\beta_2(P_n)\le 2$. Moreover, the 2-set-metric dimension $\beta_2(P_n)=2$. Indeed, if $S=\{v_i\}$ for some $1\le i\le n$, then $\mathcal{D}(v_i)=(0)=\mathcal{D}(v_i,v_j)$ for any $j\neq i$, $j=1,\dots,n$.

(iii) Consider then the complete graph $\mathcal{K}_4$ of Fig. 1(b). We will show that a set $S\neq V$ cannot be a 2-resolving set. Without loss of generality, say $v_4\notin S$ for some 2-resolving set S. Notice that if we add vertices to a 2-resolving set, it remains 2-resolving. Hence we may assume that $S=\{v_1,v_2,v_3\}$. Since $\mathcal{D}(v_2)=(1,0,1)=\mathcal{D}(v_2,v_4)$, the set S is not 2-resolving. It follows that $\beta_2(\mathcal{K}_4)=4$. By the same token, $\beta_2(\mathcal{K}_n)=n$ for all complete graphs $\mathcal{K}_n$, $n\ge 3$. This example shows that a 2-resolving set must not be confused with so-called *doubly* resolving set which is discussed, for instance, in [4] – there it is shown that the smallest doubly resolving set in $\mathcal{K}_n$ equals $n-1$.

In this paper, we consider ℓ-resolving sets in two infinite families of graphs, namely, in the two dimensional grid graphs $P_p\Box P_q$ and the n-dimensional binary hypercubes $\mathbb{F}^n$. For the usual (1-)resolving set, it has been shown that the metric dimension of the two dimensional grid graph equals two [10]. Section 2 shows that we can determine the 2-set-metric dimension in the grid graph using a helpful geometric flavour of the problem. In Section 3, we consider ℓ-resolving sets in the binary hypercubes $\mathbb{F}^n$. For the usual (1-)resolving sets it is known that $\beta_1(\mathbb{F}^n)\le n$ [5] and, asymptotically [14],

$$\lim_{n\to\infty}\beta_1(\mathbb{F}^n)\cdot\frac{\log n}{n}=2.$$

2. On ℓ-resolving sets in a grid graph

In this section, we find the 2-set-metric dimension of the grid graph $P_p\Box P_q$ and show that the only ℓ-resolving set for $3\le\ell\le pq$ is the whole set of vertices $S=P_p\times P_q$. Recall that the path in Example 2(ii) can be interpreted as $P_n\Box P_1$ where P_1 consists of a single vertex.

Theorem 3. *Let $p,q\ge 2$ be integers. Then we have $\beta_2(P_p\Box P_q)=\min\{p,q\}+2$.*

Proof. First we consider the lower bound $\beta_2(P_p\Box P_q)\ge\min\{p,q\}+2$. Let S be any 2-resolving set in the graph $P_p\Box P_q$. Denote $P_p=(v_1,\dots,v_p)$ and $P_q=(w_1,\dots,w_q)$. The distance between two vertices (v_i,w_j) and $(v_{i'},w_{j'})$ of $P_p\times P_q$ equals

$$|i-i'|+|j-j'|. \tag{1}$$

First we show that all the *corners* (v_1,w_1), (v_p,w_1), (v_1,w_q) and (v_p,w_q) necessarily belong to the 2-resolving set S. Assume to the contrary that $(v_1,w_1)\notin S$ (proceed analogously with the other corners). Consider now two sets $X=\{(v_2,w_2)\}$ and $Y=\{(v_1,w_1),(v_2,w_2)\}$. By (1), we see that any vertex in $P_p\Box P_q$ apart from (v_1,w_1) has shorter (or equal) distance to (v_2,w_2) than to (v_1,w_1). Therefore, for any element $s\in S$, we get $d(s,Y)=d(s,(v_2,w_2))=d(s,X)$. Consequently, $\mathcal{D}(X)=\mathcal{D}(Y)$, which is a contradiction, and we are done.

If $p=2$ or $q=2$, this already gives the claim $\beta_2(P_p\Box P_q)\ge 4$, so assume from now on that $p,q\ge 3$. We denote the *rows* (which are not intersecting the corners) by $R_k=\{(v_i,w_k)\mid i=1,\dots,p\}$, where $k=2,\dots,q-1$, and *columns* by $I_h=\{(v_h,w_j)\mid j=1,\dots q\}$, where $h=2,\dots,p-1$. Denote the *cross* (without the center (v_h,w_k)) by $C_{h,k}=(R_k\cup I_h)\setminus\{(v_h,w_k)\}$. We need the following fact:

- **Fact 1.** There exists at least one element of S in any cross $C_{h,k}$ where $h=2,\dots,p-1$ and $k=2,\dots,q-1$. In order to prove this, let us consider the sets $X=\{(v_h,w_{k+1}),(v_h,w_{k-1})\}$ and $Y=\{(v_{h-1},w_k),(v_{h+1},w_k)\}$. There must be an element of S in the cross $C_{h,k}$ if S is a 2-resolving set, since any vertex u outside the cross has $d(u,X)=d(u,Y)$. Indeed, suppose