Information Processing Letters 116 (2016) 706–709

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

An extension of Hall's theorem for partitioned bipartite graphs ☆

Ehab Morsy

Department of Mathematics, Suez Canal University, Ismailia 41522, Egypt

ARTICLE INFO

Article history:
Received 5 October 2015
Received in revised form 6 May 2016
Accepted 15 June 2016
Available online 17 June 2016
Communicated by J.L. Fiadeiro

Keywords:
Hall's theorem
Maximum matching
Perfect matching
Maximum flow
Combinatorial problems

ABSTRACT

Let $G = (X, Y, E)$ be a bipartite graph with bipartition X and Y and edge set E such that X is partitioned into a set of k pairwise disjoint subsets $X_1, X_2, \dots, X_k$. For any sequence $n_1, n_2, \dots, n_k$ of natural numbers with $n_i \le |X_i|$ for all i, we prove a necessary and sufficient condition for the existence of a semi-perfect matching in G, a matching that includes, for each i, at least n_i edges that are incident to vertices from X_i. Clearly, this is equivalent to Hall's theorem in the case where $n_i = |X_i|$ for all i.

1. Introduction

Let $G = (X, Y, E)$ be a bipartite graph with bipartition X and Y and edge set E such that an edge (x, y) in E means that vertex $x \in X$ can be assigned to vertex $y \in Y$. Let n and m denote the cardinalities of the sets of vertices and edges in G, respectively. A *matching* in graph G is a set of pairwise non-adjacent edges, and a *maximum* matching in G is a matching that contains the maximal number of edges.

The problem of finding a maximum matching in G can be reduced to the maximum flow problem as follows. We first add a source s with edges to all vertices in X, and a sink t with edges from all vertices in Y. Next, we assign a unit capacity to each edge of the resulting graph, and then compute a maximal flow from s to t. It is easy to verify that the set of all edges with nonzero flow from X to Y forms a maximum matching in G. Therefore, algorithms designed for the maximum flow problem can be used to solve the maximum matching problem in bipartite graphs. Thus, Dinic's algorithm [5] can be used to find a maximum matching in G in $O(\sqrt{n}m)$ time. Different algorithms of the same running time are proposed for finding maximum matching in general graphs [2,7,12]. By adapting Dinic's algorithm to the maximum matching problem in bipartite graphs, Hopcroft and Karp [10] improved the time complexity to $O(\sqrt{\kappa}m)$, where κ denotes the cardinality of a maximum matching in G. Feder and Motwani [6] applied Dinic's algorithm after compressing G, reducing the number of edges by about a factor of $\log n$. They proved a running time of $O(\sqrt{n}m^*)$, where m^* is the cardinality of the set of edges in the compressed graph. Based on the fast matrix multiplication algorithm, Mucha and Sankowski [13] developed a randomized algorithm with time complexity $O(n^{2.38})$ for the maximum matching problem in bipartite graphs.

A maximum matching M in G is called *perfect matching* if, for every $x \in X$, there exists an edge that incident to x in M. Hall [4] proposed the following famous theorem that provides a necessary and sufficient condition for the existence of a perfect matching in bipartite graphs. For a subset A of X, define $N_G(A)$ to be the set of all vertices $y \in Y$ that are endpoints of edges with at least one

☆ This research was partially supported by Alexander von Humboldt Foundation.

E-mail address: ehabmorsy@science.suez.edu.eg.

http://dx.doi.org/10.1016/j.ipl.2016.06.005
0020-0190/

endpoint in A, and let $|A|$ and $|N_G(A)|$ denote the cardinalities of A and $N_G(A)$, respectively.

Theorem 1 *(Hall's theorem). There exists a perfect matching in the bipartite graph $G = (X, Y, E)$ if and only if $|N_G(A)| \geq |A|$ for every subset A of X.* □

Several applications and extensions of Hall's theorem have been considered in the literatures. The rest of this section is devoted for summarizing some of these extensions.

Let $\mathcal{S} = \{S_1, S_2, S_3, \ldots\}$ be a family of subsets of a given set A. A system of distinct representatives of $\mathcal{S}$ is an indexed set $\{a_1, a_2, a_3, \ldots\}$ of distinct elements of A such that $a_i \in S_i$ for all i. Hall [8] showed that there exists a system of distinct representatives of $\mathcal{S}$ if the union of any k distinct subsets from $\mathcal{S}$ contains at least k distinct elements, for every finite k. He also showed that this condition is sufficient if the number of subsets is finite (i.e., $\mathcal{S}$ is finite). Afterward, Hall [9] proved that the above condition is sufficient if every subset S_i in $\mathcal{S}$ is finite.

Let $G = (X, Y, E)$ be a bipartite graph such that $X = \cup_{1 \leq i \leq k} A_i$ and $Y = \cup_{1 \leq i \leq k} B_i$, where $\{A_1, A_2, \ldots, A_k\}$ and $\{B_1, B_2, \ldots, B_k\}$ are two families of finite sets. Pinelis [14] proved a necessary and sufficient condition for the existence of a perfect matching M in G such that, for each $i = 1, 2, \ldots, k$, all vertices of A_i are assigned in M to vertices from B_i.

A $(1, k)$-complete matching from X to Y in bipartite graph $G = (X, Y, E)$ is a subgraph of G in which each vertex in X is adjacent to exactly k distinct vertices from Y and each vertex in Y is adjacent to at most one vertex from X. Longani [11] proved a necessary and sufficient condition for the existence of $(1, k)$-complete matching in bipartite graphs for any natural number k.

Bokal et al. [3] proved a characterization of bipartite graph $G = (X, Y, E)$ that admits a spanning subgraph in which the degrees of vertices in X and Y satisfy specified upper and lower bounds, respectively. Formally, for any two mappings $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$, a set $E' \subseteq E$ of edges is an (f, g)-quasi-matching of G if every element y of Y has at least $g(y)$ incident edges from E', and every element x of X has at most $f(x)$ incident edges from E', where $\mathbb{N}$ denotes the set of natural numbers. Bokal et al. [3] proved necessary and sufficient conditions for the existence of (f, g)-quasi-matching in G.

Another generalization of Hall's theorem was obtained by Aharoni and Haxell [1] for hypergraphs (a graph in which each edge may connect more than two vertices). A matching in the hypergraph is a set of pairwise disjoint edges. They proved a necessary and sufficient condition for the existence of a system of pairwise disjoint representatives for a family of hypergraphs.

Now, consider the situation in which we have a set of projects each of them consists of a finite set of tasks, and a set of machines for carrying out these tasks. Each machine can process at most one task at a time, and a task can be processed by a machine if they satisfy some criteria (e.g., the processing cost is bounded by a given upper bound). The proposed plan may require a certain portion (number of tasks) of each project to be accomplished by the end of the first stage. It is easy to see that this problem can be formulated as a matching problem in a bipartite graph. Up to our knowledge, this model cannot be reduced to one of the known versions of matching problems. In this note, we prove a necessary and sufficient condition for the existence of a desired matching in this model. We also show that this matching (if one exists) can be computed in polynomial time.

2. Hall's theorem for partitioned bipartite graphs

Define a *partitioned bipartite graph* $(G = (X, Y, E), \mathcal{X}, \mathcal{N})$ in which $G = (X, Y, E)$ is a bipartite graph, $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ is a partition of X, and $\mathcal{N}$ is a sequence of k natural numbers $n_1, n_2, \ldots, n_k$ with $n_i \leq |X_i|$ for all i. A matching M in a partitioned bipartite graph $(G = (X, Y, E), \mathcal{X}, \mathcal{N})$ is called a *semi-perfect matching* if, for every $i = 1, 2, \ldots, k$, at least n_i edges in M are incident to vertices from the set X_i. In this section we prove a necessary and sufficient condition for the existences of a semi-perfect matching in partitioned bipartite graphs. We assume without loss of generality that each vertex x in X is adjacent to at least one vertex in Y since otherwise we can simply delete x from X without affecting the existence of a semi-perfect matching in $(G = (X, Y, E), \mathcal{X}, \mathcal{N})$.

For a subset $A \subseteq X$, let $d^A_{(G,\mathcal{X},\mathcal{N})}$ denote the maximum number of vertices in A that might not be included in any semi-perfect matching in $(G = (X, Y, E), \mathcal{X}, \mathcal{N})$, i.e., $d^A_{(G,\mathcal{X},\mathcal{N})} = \sum_{1 \leq i \leq k} \min\{|A \cap X_i|, |X_i| - n_i\}$.

The following theorem is an extension of Hall's theorem in partitioned bipartite graphs. The main idea of the proof is similar to that of the original Hall's theorem.

Theorem 2. *There exists a semi-perfect matching in a partitioned bipartite graph $(G = (X, Y, E), \mathcal{X}, \mathcal{N})$ if and only if*

$$|N_G(A)| \geq |A| - d^A_{(G,\mathcal{X},\mathcal{N})} \tag{1}$$

for every subset A of X.

Proof. If the partitioned bipartite graph $(G = (X, Y, E), \mathcal{X}, \mathcal{N})$ has a semi-perfect matching, then it is easy to verify that it satisfies Condition (1).

Now, assume that the partitioned bipartite graph $(G = (X, Y, E), \mathcal{X}, \mathcal{N})$ satisfies Condition (1) and we want to show that it has a semi-perfect matching. We prove the theorem by induction on the cardinality of the set X. If X contains one vertex x (i.e., $X = \{x\}$), then the theorem is trivially true since x is adjacent to at least one vertex in Y by the assumptions. Assume that the theorem is true for $|X| \leq n$, and consider a bipartite graph G with $|X| = n + 1$. We consider the following two cases.

Case 1. In the first case, $|N_G(A)| \geq |A| - d^A_{(G,\mathcal{X},\mathcal{N})} + 1$ holds for every subset $A \subset X$. Then we choose any vertex $x \in X$, say from a subset X_i in $\mathcal{X}$, and any $y \in N_G(\{x\})$. Note that y is well defined since the set $N_G(\{x\})$ is nonempty by the assumptions. Let $G' = (X', Y', E')$ be the bipartite graph with bipartition $X' = X - \{x\}$ and $Y' = Y - \{y\}$, and whose set of edges E' is the same as those in E after deleting all edges incident to x and y. Define a partition $\mathcal{X}' = \{X'_1, X'_2, \ldots, X'_k\}$ of X' such that