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Degree condition for completely independent spanning trees ☆

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ABSTRACT

Let $T_1, T_2, \dots, T_k$ be spanning trees of a graph G. For any two vertices u, v of G, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \dots, T_k$ are completely independent. Araki showed that a graph G on $n \geq 7$ vertices has two completely independent spanning trees if the minimum degree $\delta(G) \geq n/2$. In this paper, we give a generalization: a graph G on $n \geq 4k - 1$ vertices has k completely independent spanning trees if the minimum degree $\delta(G) \geq \frac{k-1}{k}n$. In fact, we prove a stronger result.

1. Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The *vertex set* and the *edge set* of G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, the *neighbor set* $N_G(v)$ is the set of vertices adjacent to v, $d_G(v) = |N_G(v)|$ is the *degree* of v. $\delta(G) = \min\{d_G(v) : v \in V(G)\}$ is the *minimum degree* of G. For a subset $U \subset V(G)$, the subgraph induced by U is denoted by $G[U]$ and $G - U$ is subgraph induced by $V(G) \setminus U$. If $U = \{u\}$ then we shall use $G - u$ instead of $G - \{u\}$. An isomorphism of graphs G and H is a bijection between the vertex sets of G and H such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. If an isomorphism exists between two graphs, then the graphs are called isomorphic and we write $G \cong H$.

Let x, y be two vertices of G. An (x, y)-*path* is a path with the two ends x and y. Two (x, y)-paths P_1, P_2 are *openly disjoint* if they have no common edge and no common vertex except for the two ends x and y. Let $T_1, T_2, \dots, T_k$ be spanning trees in a graph H. For any two vertices u, v of H, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \dots, T_k$ are completely independent spanning trees (CISTs) in G. Readers are referred to [3,5,7] for completely independent spanning trees and its applications.

The concept of completely independent spanning trees was proposed by Hasunuma [3]. In [3], Hasunuma gave a characterization for CISTs and proved that the underlying graph of a k-connected line digraph always contains k CISTs. It is well known [8,10] that every $2k$-edge-connected graph has k edge-disjoint spanning trees. Motivated by this, Hasunuma [4] conjectured that every $2k$-connected graph has k CISTs. However, Péterfalvi [9] disproved the conjecture by constructing a k-connected graph, for each $k \geq 2$, which does not have two CISTs. In [3], Hasunuma studied CISTs in some special graphs. Recently, Araki [1] provided a new characterization of the existence of k CISTs and showed that a graph G has two CISTs if $\delta(G) \geq n/2$ or G is a square of a 2-connected graph. It is interesting to note that these two well-known conditions are also sufficient for a graph to be Hamiltonian. So, Araki [1] asked that whether other sufficient conditions for a graph to be Hamiltonian also imply the existence of two CISTs. In [2], Fan et al. confirmed that the well-known Ore's condition

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also implies the existence two CISTs. In this paper, we focus on minimum degree conditions and generalize the existence of two CISTs to k CISTs in the following theorem.

Theorem 1.1. *Let G be a graph on $n \geq 4k-1 \geq 7$ vertices. If $\delta(G) \geq \frac{k-1}{k}n$ then G has k completely independent spanning trees.*

When $k=2$, it is exactly the Araki's result in [1] (see Theorem 2.2). In Section 2, some preliminaries are given. In Section 3, a stronger version of Theorem 1.1 will be proved.

We noticed that Hasunuma [6] independently obtained a similar result about the existence of k CISTs, which states that for any graph G on $n \geq 7$ vertices and for any integer k with $3 \leq k \leq n/2$, if $\delta(G) \geq n-k$ then G has $\lfloor \frac{n}{k} \rfloor$ CISTs. In fact, this is basically equivalent to our Theorem 1.1.

2. Preliminaries

In this section, the characterization given by Araki [1] will be stated. Let $(V_1, V_2, \ldots, V_k)$ be a partition of the vertex set $V(G)$ and, for $i \neq j$, $B(V_i, V_j, G)$ be a bipartite graph with the edge set $\{uv | uv \in E(G), u \in V_i$ and $v \in V_j\}$. If the graph G is clear from the context, we may use $B(V_1, V_2)$ instead of $B(V_1, V_2, G)$. A partition $(V_1, V_2, \ldots, V_k)$ is called a CIST-partition of G if it satisfies the following two conditions:

(1) for $i = 1, 2, \ldots, k$, the induced subgraph $G[V_i]$ is connected and

(2) for any $i \neq j$, the bipartite graph $B(V_i, V_j)$ has no tree components, that is, every connected component H of $B(V_i, V_j)$ satisfies $|E(H)| \geq |V(H)|$.

Araki [1] proved the existence of k CISTs is in fact equivalent to the existence of a CIST-partition.

Theorem 2.1 *([1]). A connected graph G has k completely independent spanning trees if and only if there is a CIST-partition $(V_1, \ldots, V_k)$ of $V(G)$.*

By using Theorem 2.1, Araki [1] proved that Dirac's condition yields a CIST-partition (V_1, V_2) of $V(G)$ and obtained the following result.

Theorem 2.2 *([1]). Let G be a graph with $n \geq 7$ vertices. If $\delta(G) \geq n/2$, then G has two completely independent spanning trees.*

In this paper, we will show that Dirac's condition in fact implies a balanced version of CIST-partition. A partition $(V_1, V_2, \ldots, V_k)$ is *balanced* if $||V_i| - |V_j|| \leq 1$ for all $i \neq j$. In order to state the proof clearly, we give the following lemma first.

Lemma 2.3. *Let G be a graph and $\delta(G) \geq n/2$, (V_1, V_2) is a balanced partition of $V(G)$. If $d_{B(V_1,V_2)}(v) \geq 2$ for all $v \in V(G)$, then there exist two subsets $X \subseteq V_1$ and $Y \subseteq V_2$ such that $|X| = |Y|$, $B(X, Y)$ is connected and every component of $B(V_1 \cup Y \setminus X, V_2 \cup X \setminus Y)$ contains a cycle except for the component containing $B(X, Y)$.*

Proof. Let $x_1 y_1$ be the edge of $B(V_1, V_2)$ such that $x_1 \in V_1$, $y_1 \in V_2$ and $\min\{|N(x_1) \cap V_2|, |N(y_1) \cap V_1|\}$ is minimum and subject to this $|N(x_1) \cap V_2| + |N(y_1) \cap V_1|$ is minimum. Let $X_1 = \{x_1\}$ and $Y_1 = \{y_1\}$. We will show that X_1 and Y_1 are the desire two subsets in most cases. In fact, we will give a characterization of G under the assumption that X_1 and Y_1 are not the desire two subsets.

Claim 1. $\min\{|N(x_1) \cap V_2|, |N(y_1) \cap V_1|\} = 2$.

Suppose, to the contrary, that $\min\{|N(x_1) \cap V_2|, |N(y_1) \cap V_1|\} \geq 3$. Then by the choice of $x_1 y_1$, $\delta(B(V_1, V_2)) \geq 3$. It follows that $\delta(B(V_1, V_2) - \{x_1, y_1\}) \geq 2$. Thus every component of $B(V_1, V_2) - \{x_1, y_1\}$ has a cycle, which implies X_1, Y_1 are the desire two subsets, a contradiction to the assumption and the claim is proved.

By Claim 1, without loss of generality, we may assume $N(x_1) \cap V_2 = \{y_1, y_2\}$. Let $U_1 = V_1 \cup Y_1 \setminus X_1$ and $U_2 = V_2 \cup X_1 \setminus Y_1$. As $\delta(G) \geq \lceil n/2 \rceil \geq |V_1|$,

$$
\begin{aligned}
|V_1 \setminus (N(x_1) \cup \{x_1\})| &= |V_1| - 1 - |N(x_1) \cap V_1| \\
&\leq |V_1| - 1 - (\lceil n/2 \rceil - 2) \leq 1. \quad (1)
\end{aligned}
$$

Claim 2. $V_1 \setminus \{x_1\} \nsubseteq N(x_1)$.

Suppose, to the contrary, that $V_1 \setminus \{x_1\} \subseteq N(x_1)$. Then x_1 is adjacent to every vertex of U_1 in $B(U_1, U_2)$. Also, by the assumption $d_{B(V_1,V_2)}(v) \geq 2$ for all $v \in V(G)$, $N(v) \cap U_2 \neq \emptyset$ for all $v \in U_2$. Thus $B(U_1, U_2)$ is connected. Then X_1, Y_1 are the desire two subsets, a contradiction to the assumption. The claim is proved.

By Claim 2 and (1), we may assume that $V_1 \setminus (\{x_1\} \cup N(x_1)) = \{x_2\}$. Then we have the following claim.

Claim 3. $x_2 y_2 \in E(G)$.

For any vertex $z \in V_2 \setminus \{y_1, y_2\}$, as $|N(z) \cap V_1| \geq 2$, z is adjacent to some vertex in $N(x_1) \cap V_1$. It follows that $B(U_1, U_2) - \{x_2, y_2\}$ is connected. If $x_2 y_2 \notin E(G)$ then $N(x_2) \cap (V_2 \setminus \{x_1, y_2\}) \neq \emptyset$ and $N(y_2) \cap (V_1 \setminus \{y_1, x_2\}) \neq \emptyset$. Thus $B(U_1, U_2)$ is connected and X_1, Y_1 are the desire two subsets, a contradiction to the assumption. The claim is proved.

Moreover, if $|N(x_2) \cap V_2| \geq 3$ or $|N(x_2) \cap V_2| = 2$ and $N(x_2) \cap V_2 \neq \{y_1, y_2\}$, then it is easy to see that $B(U_1, U_2)$ is connected and X_1, Y_1 are the desire two subsets, a contradiction again. So $N(x_2) \cap V_2 = \{y_1, y_2\}$. Similarly, $N(y_2) \cap V_1 = \{x_1, x_2\}$. Thus $|N(x_2) \cap V_2| + |N(y_2) \cap V_1| = 4$. By the choice of $x_1 y_1$, we also have $|N(x_1) \cap V_2| + |N(y_1) \cap V_1| = 4$. This implies $\{x_1, x_2, y_1, y_2\}$ induces a component of $B(V_1, V_2)$. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. It is easy to see that X and Y are the desire two subsets. □

3. Main results

In this section, we show that a graph G with large minimum degree admits a balanced CIST-partition, which will be used to find k completely independent spanning trees of G.