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A note on graph proper total colorings with many distinguishing constraints

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ABSTRACT

A proper edge-coloring of a simple graph G is called a vertex distinguishing edge-coloring if for any two distinct vertices u and v of G, the set of the colors assigned to the edges incident to u differs from the set of the colors assigned to the edges incident to v. We extend such distinguishing edge colorings to proper total colorings with many distinguishing constraints and color several kinds of graphs totally with the least number of colors such that the graphs admit proper total colorings having at least four distinguishing constraints.

1. Introduction and concepts

Labeled graphs are becoming an increasingly useful family of mathematical models for a broad range of applications, such as *time tabling and scheduling*, *frequency assignment*, *register allocation*, *computer security* and so on. In [2], Burris and Schelp introduced a proper edge-coloring of a simple graph G that is called a *vertex distinguishing edge-coloring* (vdec) if for any two distinct vertices u and v of G, the set of the colors assigned to the edges incident to u differs from the set of the colors assigned to the edges incident to v. The minimum number of colors required for all vertex distinguishing colorings of G is denoted by $\chi'_s(G)$. Let $n_d = n_d(G)$ denote the number of all vertices of degree d in G. It is clear that $\binom{\chi'_s(G)}{d} \geq n_d$ for all d with respect to $\delta(G) \leq d \leq \Delta(G)$. A graph is *vertex-distinguishably edge-colorable*, or a *vdec graph*, if it contains no more than one isolated vertex and no isolated edges [2]. Burris and Schelp [2] presented the following conjecture:

Conjecture 1. *Let G be a vdec graph, and let k be the minimum integer such that $\binom{k}{d} \geq n_d$ for $\delta(G) \leq d \leq \Delta(G)$. Then $\chi'_s(G) = k$ or $k+1$.*

A weak version of a vdec introduced in [5] is the adjacent vertex distinguishing edge-coloring (avdec). Zhang et al. [5] asked that for every edge xy of G, the set of the colors assigned to the edges incident to x differs from the set of the colors assigned to the edges incident to y in an avdec, and used the notation $\chi'_{as}(G)$ to denote the least number of k colors required for which G admits an avdec. They proposed a conjecture: Every simple graph G having no isolated edges and at most one isolated vertex and being not a cycle of five vertices holds $\chi'_{as}(G) \leq \Delta(G) + 2$. Successively, Zhang et al. [6] investigated the chromatic number $\chi''_{as}(G)$ and conjectured: $\chi''_{as}(G) \leq \Delta(G) + 3$ for every simple graph G. Surprisingly, it is very difficult to settle down the above three conjectures, even to verify them for simpler graphs [1]. Many distinguishing types of colorings are investigated [4].

We use standard notation and terminology of graph theory. Graphs mentioned here are simple, undirected and finite. The shorthand notation $[\alpha, \beta]$ is used to denote a set $\{\alpha, \alpha+1, \dots, \beta\}$ in the following, where integers α, β

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hold $\beta > \alpha \geq 0$. The set of vertices adjacent to a vertex x of a graph H is denoted by $N(x)$. We call H *simple* if $\deg_H(x) = |N(x)|$ for every $x \in V(H)$, where $\deg_H(x)$ is the degree of x. Let f be a proper total coloring of a simple graph G. The colors of neighbors of a vertex u of G and the colors of edges incident to u form four color sets $C(f,u) = \{f(ux) : x \in N(u)\}$, $C\langle f,u\rangle = \{f(x) : x \in N(u)\} \cup \{f(u)\}$, $C[f,u] = C(f,u) \cup \{f(u)\}$, and $C_2[f,u] = C(f,u) \cup C\langle f,u\rangle$. Notice that $\deg_G(u) + 1 \leq |C_2[f,u]|$. These color sets give rise to a couple of distinguishing total colorings. We have two sets of *distinguishing constraints* in the following.

For the purpose of convenience, let $\mathcal{F}_{3s}(n)$ be the set of simple graphs with $n \geq 3$ vertices and no isolated edges as well as at most one isolated vertex. The notation $G(h,u) \neq G(h,v)$ means that four distinguishing constraints $C(f,u) \neq C(f,v)$, $C\langle f,u\rangle \neq C\langle f,v\rangle$, $C[f,u] \neq C[f,v]$ and $C_2[f,u] \neq C_2[f,v]$ exist, simultaneously.

Definition 1. For a simple graph $G \in \mathcal{F}_{3s}(n)$ let f be a proper total k-coloring from $V(G) \cup E(G)$ to $[1,k]$. We call f an (8)-*distinguishing total coloring* (8-vdtc) if it holds $G(h,u) \neq G(h,v)$ for distinct vertices $u, v \in V(G)$. The minimum number of k colors required for which G admits an 8-vdtc is denoted as $\chi''_{8s}(G)$. A 4-*avdtc* is a proper total coloring holding $G(h,u) \neq G(h,v)$ for every edge $uv \in E(G)$; and the chromatic number $\chi''_{4as}(G)$ is the minimum number of k colors required for which G admits a 4-avdtc.

Clearly, some simple graphs do not admit 8-vdtcs or 4-avdtcs, for example, such are complete graphs.

2. Graphs having 8-vdtcs or 4-avdtcs

Lemma 1. *A graph $G \in \mathcal{F}_{3s}(n)$ admits a total coloring f with $C\langle f,u\rangle \neq C\langle f,v\rangle$ for distinct vertices $u, v \in V(G)$ (resp. for every edge $uv \in E(G)$) if and only if $N(u) \cup \{u\} \neq N(v) \cup \{v\}$ for distinct $u, v \in V(G)$ (resp. every edge $uv \in E(G)$).*

Proof. To show the proof of 'if', we take a total coloring f with $C\langle f,u\rangle \neq C\langle f,v\rangle$ for distinct vertices $u, v \in V(G)$. If $uv \in E(G)$, $C\langle f,u\rangle \neq C\langle f,v\rangle$ means that $\{f(x) : x \in N(u)\} \setminus \{f(u), f(v)\} \neq \{f(x) : x \in N(v)\} \setminus \{f(u), f(v)\}$, and furthermore $N(u) \cup \{u\} \neq N(v) \cup \{v\}$. If $uv \notin E(G)$, $C\langle f,u\rangle \neq C\langle f,v\rangle$ means $N(u) = N(v)$, or $N(u) \neq N(v)$. No matter which one of two cases occurs, we have $N(u) \cup \{u\} \neq N(v) \cup \{v\}$.

To show the proof of 'only if', it is straightforward to provide a total coloring h of a graph G with $N(u) \cup \{u\} \neq N(v) \cup \{v\}$ for distinct $u, v \in V(G)$ (including every edge $uv \in E(G)$). In fact, we can set a bijection h from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$. Clearly, $C\langle h,u\rangle \neq C\langle h,v\rangle$ for distinct $u, v \in V(G)$ (including every edge $uv \in E(G)$) by the choice of G. □

Theorem 2. *A tree T with $n \geq 3$ vertices and $n_2(T) = 0$ holds*

$$n_1(T) \leq \chi''_{8s}(T) \leq n_1(T) + 1. \tag{1}$$

The bounds are sharp.

Proof. First of all, we consider that the tree T is a star $K_{1,m}$, where $V(K_{1,m}) = \{u, v_i : i \in [1,m]\}$ and $E(K_{1,m}) = \{uv_i : i \in [1,m]\}$. We define a total coloring f of $K_{1,m}$ as: $f(v_i) = i$ for $i \in [1,m]$, $f(u) = m+1$, and $f(uv_j) = j-1$ for $j \in [2,m]$, and $f(uv_1) = m$. Clearly, f is an 8-vdtc, and $\chi''_{8s}(K_{1,m}) \leq \max f(V(K_{1,m}) \cup E(K_{1,m})) = n_1(K_{1,m}) + 1$. On the other hand, $\chi''_{8s}(K_{1,m}) \geq \Delta(K_{1,m}) + 1 = n_1(K_{1,m}) + 1$. Thereby, $\chi''_{8s}(K_{1,m}) = n_1(K_{1,m}) + 1$.

Assume that T is a tree having $n_2(T) = 0$ and diameter at least three. So, there are at least two vertices w, w' of T such that w is adjacent to $\deg_T(w) - 1$ leaves of T, and w' is adjacent to $\deg_T(w') - 1$ leaves of T. We call such vertices w, w' as the *end-nodes* of T. In fact, each longest path of T contains at least two end-nodes. Suppose that w_0 is an end-node with the smallest degree among the end-nodes of T. Notice that $\deg_T(w_0) \geq 3$ according to the theorem's hypothesis. Let $N(w_0) = \{u_0, u_i : i \in [1,t]\}$, where $t = \deg_T(w_0) - 1 \geq 2$, and $\deg_T(u_i) = 1$ for $i \in [1,t]$, $\deg_T(u_0) \geq 3$. We have a tree H obtained by deleting $u_1, u_2 \ldots, u_t$ from T such that $n_1(H) + t - 1 = n_1(T)$. Notice that H holds $n_2(H) = 0$ and $|H| < |T|$. By induction hypothesis, H admits an 8-vdtc g having $n_1(H) \leq \chi''_{8s}(H) \leq n_1(H) + 1$.

Case 1. $|g(V(H) \cup E(H))| = n_1(H)$. We extend the 8-vdtc g to a total coloring g' of T as: $g'(z) = g(z)$ for $z \in V(H) \cup E(H) \subset V(T) \cup E(T)$; $g'(w_0u_i) = n_1(H) + i$ for $i \in [1,t]$; $g'(u_i) = n_1(H) + i - 1$ for $i \in [2,t]$; and $g'(u_1) = n_1(H) + t$. Clearly, g' is an 8-vdtc of T such that $n_1(T) \leq \chi''_{8s}(T) \leq n_1(T) + 1$ since $n_1(H) + t = n_1(T) + 1$.

Case 2. $|g(V(H) \cup E(H))| = n_1(H) + 1$. There exists a color k_0 which does not appear at any leaf of H, that is, no edge w_jl_j with $\deg_H(w_j) \geq 2$ and $\deg_H(l_j) = 1$ is colored with the color k_0. We define another total coloring h of T in the following Case 2.1 and Case 2.2. Let $M = (V(T) \setminus \{u_i : i \in [1,t]\}) \cup (E(T) \setminus \{w_0u_i : i \in [1,t]\})$.

Case 2.1. $g(w_0) \neq k_0$. We set $h(w_0u_1) = k_0$, $h(w_0u_i) = n_1(H) + i$ for $i \in [2,t]$; $h(u_s) = n_1(H) + 1 + s$ for $s \in [1, t-1]$, $h(u_t) = k_0$; $h(z) = g(z)$ for $z \in M$. We check the color sets of T. Let $F(h,z) = \{C(h,z), C\langle h,z\rangle, C[h,z], C_2[h,z]\}$ for every vertex $z \in V(T)$. The notation $F(h,x) \neq F(h,y)$ for distinct $x, y \in V(T)$, in the following argument, means one of four cases $C(h,y) \neq C(h,x)$, $C\langle h,y\rangle \neq C\langle h,x\rangle$, $C[h,y] \neq C[h,x]$ and $C_2[h,y] \neq C_2[h,x]$.

(1.1) Since every vertex $z \in M$ holds $C(h,z) = C(g,z)$, $C\langle h,z\rangle = C\langle g,z\rangle$, $C[h,z] = C[g,z]$ and $C_2[h,z] = C_2[g,z]$. So $F(h,z) \neq F(h,z')$ for distinct $z, z' \in M$.

(1.2) Notice that w_0 is adjacent to u_2, $\deg_T(w_0) \geq 3$, $h(w_0u_2) = n_1(H) + 2$ and $h(u_2) = n_1(H) + 3$. Thereby, $F(h, w_0) \neq F(h,x)$ for $x \in V(T) \setminus \{w_0\}$.

(1.3) For every vertex $u_i \in N(w_0)$, $i \in [1, t-1]$, we can conclude that $F(h,u_i) \neq F(h,x)$ for $x \in V(T) \setminus \{u_i\}$, since $\deg_T(w_0) = t + 1 \geq 3$. The last vertex u_t has $C(h,u_t) = \{h(w_0u_t)\} = \{n_1(H) + t\}$, $C\langle h,u_t\rangle = \{h(w_0), k_0\}$, $C[h,u_t] = \{h(u_t), n_1(H) + t\}$ and $C_2[h,u_t] = \{h(w_0), k_0, n_1(H) + t\}$. Only the color set $C\langle h,u_t\rangle = \{h(w_0), k_0\}$ may be equal to some $C\langle h,x\rangle$ for some one-degree vertex $x \in V(T) \setminus \{w_0, u_0\}$, and we recolor u_t with $h(u_0)$ such that $C\langle h,u_t\rangle =$