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Acyclically 4-colorable triangulations

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ABSTRACT

An acyclic k-coloring of a graph is a proper vertex k-coloring such that every bichromatic subgraph, induced by this coloring, contains no cycles. A graph is acyclically k-colorable if it has an acyclic k-coloring. In this paper, we prove that every acyclically 4-colorable triangulation with minimum degree more than 3 contains at least four odd-vertices. Moreover, we show that for an acyclically 4-colorable triangulation with minimum degree 4, if it contains exactly four odd-vertices, then the subgraph induced by its four odd-vertices is triangle-free and claw-free.

1. Introduction

All graphs considered in this paper are simple and finite. For a graph G, we denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ the set of vertices, the set of edges, the *minimum degree* and *maximum degree* of G, respectively. For a vertex u of G, $d_G(u)$ is the degree of u in G. We call u a *k-vertex* if $d_G(u) = k$. If k is an odd number, we say u to be an *odd-vertex*, and otherwise an *even-vertex*. If $d_G(u) > 0$, then each adjacent vertex of u is called a *neighbor* of u. The set of all neighbors of u in G is denoted by $N_G(u)$. Note that $N_G(u)$ does not include u itself. We then write $N_G[u] = N_G(u) \cup \{u\}$. For a subset $V' \subseteq V(G)$, we denote by $G[V']$ the subgraph of G induced by V'. For more notations and terminologies, we refer the reader to the book [1].

A *k-coloring* of G is an assignment of k colors to $V(G)$ such that no two adjacent vertices are assigned the same color. Let f be a k-coloring of a graph G, and H be a subgraph of G. We denote by $f(H)$ the set of colors assigned to $V(H)$ under f. For a cycle C of G, if $|f(C)| = 2$, then we call C a *bichromatic cycle* of f, or we say f *contains bichromatic cycle* C. If f does not contain any bichromatic cycle, then we call f an *acyclic k-coloring* of G. The *acyclic chromatic number* $A(G)$ of a graph G is the least number of colors needed in any acyclic coloring of G. Bounds on $A(G)$ in terms of the maximum degree $\Delta(G)$ of G include the following: $A(G) \le 4$ if $\Delta(G) = 3$ [13], $A(G) \le 5$ if $\Delta(G) = 4$ [9], $A(G) \le 7$ if $\Delta(G) = 5$ [16], which improves the result that $A(G) \le 9$ if $\Delta(G) = 5$ [11], and $A(G) \le 11$ if $\Delta(G) = 6$ [14].

The acyclic colorings was first studied by Grünbaum [13], who wrote a long paper to investigate the acyclic colorings of planar graphs. He proved that every planar graph is acyclically 9-colorable, and conjectured that five colors are sufficient. Sure enough, three years later, Borodin [2] (also see [3]) gave a proof of Grünbaum's conjecture. Indeed, five is the best possible as there exist planar graphs without acyclic 4-colorings [13]. In 1973, Wegner [21] constructed a 4-colorable planar graph G, each 4-coloring of which possesses a cycle in every bichromatic subgraph. Afterwards Kostochka and Melnikov [15] showed that graphs with no acyclic 4-coloring can be found among 3-degenerated bipartite planar graphs.

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The research on acyclically 4-colorable planar graphs always attracted much attentions. In 1999, Borodin, Kostochka, and Woodall [7] showed that planar graphs under the absence of 3- and 4-cycles are acyclically 4-colorable; In 2006, Montassier, Raspaud, and Wang [19] proved that planar graphs, without 4-,5-, and 6-cycles, or without 4-, 5-, and 7-cycles, or without 4-, 5-, and intersecting 3-cycles, are acyclically 4-colorable; In 2009, Chen and Raspaud [10] proved that if a planar graph G has no 4-, 5-, and 8-cycles, then G is acyclically 4-colorable; Borodin [4,5] showed that planar graphs without 4- and 6-cycles, or without 4- and 5-cycles are acyclically 4-colorable; Recently [6], Borodin proved that planar graphs without 4- and 5-cycles are acyclically 4-choosable.

A planar graph G is called a *triangulation* if adding any edge to G results in a nonplanar graph. The *dual* graph G^* of a plane graph G is a graph that has a vertex corresponding to each face of G, and an edge joining two neighboring faces for each edge in G. It is well-known that the dual graphs of triangulations are planar cubic 3-connected graphs. Observe that G is an acyclically 4-colorable triangulation if and only if its dual graph G^* contains three Hamilton cycles such that each edges of G^* is just contained in two of them. Since the problem of deciding whether a planar cubic 3-connected graph contains a Hamilton cycle is NP-complete [12], we can deduce that the problem of deciding whether a triangulation is acyclically 4-colorable is NP-complete. In addition, it has been shown that the acyclic 4-colorability is NP-complete for planar graphs with maximum degree 5, 6, 7, and 8 respectively and for planar bipartite graphs with the maximum degree 8 [18,17,20].

As far as we know, there are no papers investigating the acyclic 4-colorability of triangulations. Because there exist triangulations without acyclic 4-colorings, we are interested in studying what the characteristics and structures of acyclically 4-colorable triangulations are. In this paper, we prove that an acyclic 4-colorable triangulation G with $\delta(G) \geq 4$ contains at least four odd-vertices. Furthermore, for an acyclic 4-colorable triangulation with minimum degree 4 and exactly four odd-vertices, we show that the subgraph induced by its four odd-vertices is triangle-free and claw-free.

2. Main results

A k-cycle C is a cycle of length k. If k is even, we call C an even cycle, otherwise, an odd cycle. A cycle is called *separating k-cycle* in a graph embedded on the plane if it is a k-cycle such that both the interior and the exterior contain one or more vertices. A *n-wheel W_n* (or simply *wheel W*) is a graph with $n+1$ vertices ($n \geq 3$), formed by connecting a single vertex (called the *center* of W_n) to all vertices of a n-cycle. A subgraph is called a *k-wheel subgraph* of a triangulation G if it is isomorphic to a k-wheel. Obviously, when G is 4-connected, it follows that any subgraph induced by a k-vertex and all of its neighbors is a k-wheel subgraph of G.

Let W be a 4-wheel subgraph of G. The *operation of contracting 4-wheel W* on u, w of G, denoted by $\mathscr{D}_W^{u,w}(G)$, is identifying u, w and the center of W and then deleting resulting parallel edges. We denote by $\zeta_W^{u,w}(G)$ the resulting graph by conducting operation $\mathscr{D}_W^{u,w}(G)$, and (u, w) the corresponding identified vertex in $\zeta_W^{u,w}(G)$. Clearly, when $N(u) \cap N(w) \subseteq N[v]$ and $d_G(x) \geq 5$ and $d_G(y) \geq 5$, it follows that $\zeta_W^{u,w}(G)$ is still a triangulation, and

$$\begin{aligned} d_{\zeta_W^{u,w}(G)}((u,w)) &= d_G(u) + d_G(w) - 4, \\ d_{\zeta_W^{u,w}(G)}(x) &= d_G(x) - 2, \\ d_{\zeta_W^{u,w}(G)}(y) &= d_G(y) - 2, \end{aligned} \tag{1}$$

where $\{x, y\} = N_G(v) \setminus \{u, w\}$.

Lemma 2.1. *If G is a triangulation with a 4-vertex v and an acyclic 4-coloring f, then the following holds.*

(1) $|f(N_G(v))| = 3$.

(2) *There exists a pair of nonadjacent neighbors v_1, v_3 of v receiving the same color, and the set of their common neighbors is $N_G[v] \setminus \{v_1, v_3\}$.*

(3) *The neighbors of v other than v_1, v_3 receive the different colors and these two vertices are of degree at least 5.*

Proof. (1) immediately follows from the acyclicality of f. So, $N_G(v)$ contains two nonadjacent neighbors, v_1, v_3 such that $f(v_1) = f(v_3)$. Let $\{v_2, v_4\} = N_G(v) \setminus \{v_1, v_3\}$. Then, $f(v_2) \neq f(v_4)$. Suppose that there exists a vertex $u \in N(v_1) \cap N(v_3)$ but $u \notin \{v, v_2, v_4\}$, then either $v_1v_2v_3uv_1$, or $v_1vv_3uv_1$, or $v_1v_4v_3uv_1$ is a bichromatic cycle. This contradicts with f, and hence (2) holds. For (3), it can be directly deduced from (2). □

If an acyclically 4-colorable triangulation contains a 3-vertex, then the resulting graph by deleting the 3-vertex and its incident edges is still acyclically 4-colorable. Thus, in what follows we mainly consider acyclically 4-colorable triangulations without 3-vertices (i.e. with minimum degree 4), and use $\mathcal{T}4$ to denote the class of such triangulations with exactly four odd-vertices. Further, for a graph $G \in \mathcal{T}4$, we use $V^4(G)$ to denote the set of its four odd-vertices of G. According to the Euler formula, one can readily check that every $G \in \mathcal{T}4$ contains at least four 4-vertices.

For a 4-vertex v in a graph $G \in \mathcal{T}4$, if there exist a pair of nonadjacent vertices $u, w \in N_G(v)$ receiving the same color under an acyclic 4-coloring of G, such that $\zeta_W^{u,w}(G)$ is also a graph in $\mathcal{T}4$, then we refer to v as a *contractible vertex* of G. According to Lemma 2.1, we can see that for any triangulation $G \in \mathcal{T}4$, $\zeta_W^{v_1,v_2}(G)$ is still acyclically 4-colorable, where f is an arbitrary acyclic 4-coloring of G, v is a 4-vertex, and $v_1, v_2 \in N_G(v)$ satisfying $f(v_1) = f(v_2)$. Moreover, let $\{v_3, v_4\} = N_G(v) \setminus \{v_1, v_2\}$. If $d_G(v_3) \geq 6$ and $d_G(v_4) \geq 6$, then $\delta(\zeta_W^{v_1,v_2}(G)) \geq 4$ and v is a contractible vertex.

Theorem 2.2. *Every acyclically 4-colorable triangulation G with $\delta(G) \geq 4$ contains at least four odd-vertices.*

Proof. It suffices to consider $\delta(G) = 4$ because G contains at least twelve 5-vertices when $\delta(G) = 5$ by the Euler formula.

If the conclusion fails to hold when $\delta(G) = 4$, let G' be a counterexample on the fewest vertices to the the-