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L-visibility drawings of IC-planar graphs ☆

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ABSTRACT

An IC-plane graph is a topological graph where every edge is crossed at most once and no two crossed edges share a vertex. We show that every IC-plane graph has a visibility drawing where every vertex is of the form {└, ┘, ┐, ┌}, and every edge is either a horizontal or vertical segment. As a byproduct of our drawing technique, we prove that every IC-plane graph has a RAC drawing in quadratic area with at most two bends per edge.

1. Introduction

A *visibility drawing* Γ of a planar graph G maps the vertices of G into non-overlapping horizontal segments (*bars*), and the edges of G into vertical segments (*visibilities*), each connecting the two bars corresponding to its two end-vertices. Visibilities intersect bars only at their extreme points. Γ is a *strong* visibility drawing if there exists a visibility between two bars if and only if there exists an edge in G between the corresponding vertices. Every biconnected planar graph admits a strong visibility drawing (see, e.g., [15]). Conversely, if a visibility may not correspond to an edge of the graph, then Γ is a *weak* visibility drawing. Since every planar graph can be augmented to a biconnected planar graph by adding edges, every planar graph admits a weak visibility drawing.

The problem of extending visibility drawings to non-planar graphs has been first studied by Dean et al. [4]. They introduce *bar k-visibility drawings*, which are visibility drawings where each bar can see through at most k distinct bars. In other words, each visibility segment can intersect at most k bars, while each bar can be intersected by arbitrary many visibility segments. The graphs that admit a bar 1-visibility drawing are called 1-*visibile*. Brandenburg et al. and independently Evans et al. prove that 1-*planar graphs*, i.e., those graphs that can be drawn with at most one crossing per edge, are 1-visible [2,8]. They focus on a *weak* model, where there is a visibility through at most k bars if there is an edge, while the converse may not be true. In fact, having a strong model would be too restrictive in the case of bar k-visibility drawings. For example, it is easy to see that a cycle of length at least four does not admit a strong bar 1-visibility drawing [2]. In terms of readability, a clear benefit of bar k-visibility drawings is that the crossings form right angles. *Right-angle crossing (RAC) drawings* and their advantages in terms of readability have been extensively studied in the graph drawing literature (see, e.g., [7,11]). However, in a bar k-visibility drawing crossings involve bars and visibilities, i.e., vertices and edges. These crossings are arguably less intuitive than crossings between edges.

Evans et al. introduce a new model of visibility drawings, called *L-visibility drawings* [9]. Their aim is to simultaneously represent two plane *st*-graphs G_r and G_b (whose union might be non-planar). They assume a *strong* model,

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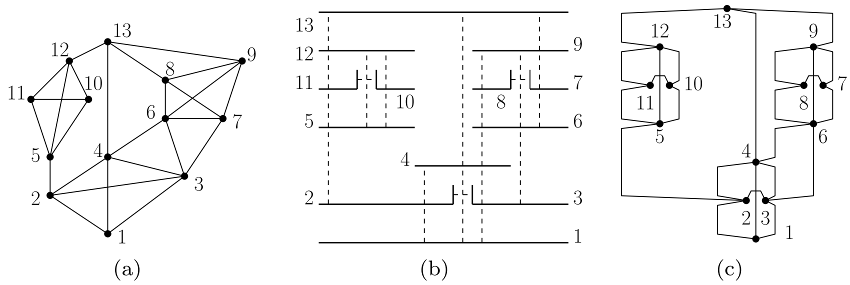

Fig. 1. (a) An IC-plane graph G. (b) A L-visibility drawing of G. (c) A RAC drawing of G with at most two bends per edge.

where each vertex is represented by a horizontal bar and a vertical bar that share an extreme point, i.e. it is an *L-shape* in the set {└, ┘, ┐, ┌}. Two L-shapes are connected by a vertical (horizontal) visibility segment if and only if there exists an edge in G_r (G_b) between the corresponding vertices, no two L-shapes cross one another, and visibilities intersect bars only at their extreme points. A clear advantage of this kind of drawing is that the only possible crossings are between vertical and horizontal visibilites, i.e., between edges of the graph. Furthermore, similar to bar k-visibilities, these crossings form right angles.

In this paper we initiate the study of *weak* L-visibility drawings of non-planar graphs. We focus on the class of graphs that admit a drawing where each edge is crossed at most once, and no two crossed edges share an end-vertex. These graphs are called *IC-planar graphs* (see Fig. 1(a) for an example). Their chromatic number is at most five [12], and they have at most $13n/4 - 6$ edges, which is a tight bound [17]. Recognizing IC-planar graphs is NP-hard [3]. Our main contribution is summarized by the following theorem, proved in Section 3. See Fig. 1(b) for an example of a drawing computed by using Theorem 1.

Theorem 1. *Every n-vertex IC-plane graph G admits a L-visibility drawing in $O(n^2)$ area, which can be computed in $O(n)$ time.*

We remark that Theorem 1 contributes to the rapidly growing literature devoted to the problem of drawing graphs that are "nearly planar" in some sense, i.e. graphs where only some types of edge crossings are allowed (for example, an edge can be crossed at most a constant number of times); see e.g., [13] for references. In particular, Brandenburg et al. have recently described a cubic-time algorithm that computes IC-planar drawings with right-angle crossings and straight-line edges [3]. However these drawings may require exponential area, which is proved to be worst-case optimal [3]. Brandenburg et al. leave as an open problem to study techniques that compute IC-planar drawings in polynomial area and with good crossing resolution [3]. We also recall that every graph admits a RAC drawing with at most three bends per edge [6], while testing whether a graph has a straight-line RAC drawing is NP-hard [1]. The following corollary follows as a byproduct of Theorem 1 (see also Fig. 1(c)).

Corollary 1. *Every n-vertex IC-plane graph G admits a RAC drawing with at most two bends per edge in $O(n^2)$ area, which can be computed in $O(n)$ time.*

2. Preliminaries

We assume familiarity with basic graph drawing concepts, see also [5].

Planarity and connectivity. A graph $G = (V, E)$ is *simple*, if it contains neither loops nor multiple edges. We consider simple graphs, if not otherwise specified. A *drawing* Γ of G maps each vertex of V to a point of the plane and each edge of E to a Jordan arc between its two end-points. We only consider *simple drawings*, i.e., drawings such that the arcs representing two edges have at most one point in common, which is either a common end-vertex or a common interior point where the two arcs properly cross. A drawing is *planar* if no two arcs representing two edges cross. A planar drawing divides the plane into topologically connected regions, called *faces*. The unbounded region is called the *outer face*. A *planar embedding* of a graph is an equivalence class of planar drawings that define the same set of faces. A graph with a given planar embedding is a *plane* graph. For a non-planar drawing, we can still talk about embedding considering that the boundary of a face may consist of portions of arcs between vertices and/or crossing points.

A graph is *biconnected* if it remains connected after removing any one vertex. A directed graph (a digraph for short) is biconnected if its underlying undirected graph is biconnected. A *topological numbering* of a digraph is an assignment, X, of numbers to its vertices such that $X(u) < X(v)$ for every edge (u, v). A graph admits a topological numbering if and only if it is acyclic. An acyclic digraph with a single source s and a single sink t is called an *st-graph*. A *plane st-graph* is an st-graph that is planar and embedded such that s and t are on the boundary of the outer face. In any st-graph, the presence of the edge (s, t) guarantees that the graph is biconnected. In the following we consider st-graphs that contain the edge (s, t), as otherwise it can be added without violating planarity. Let G be a plane st-graph, then for each vertex v of G the incoming edges appear consecutively around v, and so do the outgoing edges. Vertex s only has outgoing edges, while vertex t only has incoming edges. This particular transversal structure is known as a *bipolar orientation* [14,15]. Each face f of G is bounded by two directed paths with a common *origin* and *destination*, called the *left path* and *right path* of f.

IC-planar graphs. We recall some definitions also given in [3]. A drawing is IC-planar if each edge is crossed at most once, and any two crossed edges do not share an end-vertex. See Fig. 1(a) for an illustration. An *IC-planar*