Information and Computation 210 (2012) 40–67

Contents lists available at SciVerse ScienceDirect

Information and Computation

www.elsevier.com/locate/yinco

Temporal logics over linear time domains are in PSPACE

Alexander Rabinovich

The Blavatnik School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel

ARTICLE INFO

Article history:
Received 6 August 2010
Revised 19 June 2011
Available online 20 November 2011

ABSTRACT

We investigate the complexity of the satisfiability problem of temporal logics with a finite set of modalities definable in the existential fragment of monadic second-order logic. We show that the problem is in PSPACE over the class of all linear orders. The same techniques show that the problem is in PSPACE over many interesting classes of linear orders.

1. Introduction

A major result concerning linear-time temporal logics is Kamp's theorem [12,9,10] which states that $TL(\mathsf{Until}, \mathsf{Since})$, the temporal logic having Until and Since as the only modalities, is expressively complete for first-order monadic logic of order over the class of Dedekind-complete linear orders.

The order of natural numbers $\omega = (\mathbb{N}, <)$ and the order of the real numbers $(\mathbb{R}, <)$ are both Dedekind-complete. Another important class of Dedekind-complete orders is the class of ordinals. However, the order of the rationals is not Dedekind-complete. Stavi introduced two modalities Until_{Stavi} and Since_{Stavi} and proved that the temporal logic having the four modalities Until, Since, Until_{Stavi} and Since_{Stavi} is expressively complete for first-order monadic logic of order over the class of all linear orders [9,10].

Our concern in this paper will be with the complexity of the satisfiability problem for temporal logics over various classes of linear orders.

Sistla and Clarke [22] proved that the satisfiability problem for $TL(\mathsf{Until}, \mathsf{Since})$ over ω-models is PSPACE-complete. This proof was based on automata theoretical techniques.

Burgess and Gurevich [5] proved that $TL(\mathsf{Until}, \mathsf{Since})$ is decidable over the reals. They provided two proofs. The first involves an indirect reduction to Rabin's theorem on the decidability of the monadic second-order logic over the full binary tree [14]. The second one is based on the model-theoretical composition method. Both proofs provide algorithms of non-elementary complexity.

Reynolds [17,16] proved that the satisfiability problem for $TL(\mathsf{Until}, \mathsf{Since})$ over the reals is PSPACE-complete and that the temporal logic with only the Until modality is PSPACE-complete over the class of all linear orders. The proofs in [17,16] use temporal mosaics and are very non-trivial and difficult to grasp.

One of our objectives was to provide a simple proof of Reynolds' remarkable result [17] obtained in 1999. We also wanted a proof which can be applied to prove in a uniform way a PSPACE upper bound for other time domains. In [7], we showed that the satisfiability problem for $TL(\mathsf{Until}, \mathsf{Since})$ over the class of all ordinals is PSPACE-complete. This proof was based on automata theoretical techniques, and it is considerably simpler than Reynolds' proof of PSPACE-completeness for the satisfiability problem for $TL(\mathsf{Until}, \mathsf{Since})$ over the reals. However, the ordinals are simpler than the reals.

Cristau [6] provided a very unexpected translation from the temporal logic having the four modalities Until, Since, Until_{Stavi} and Since_{Stavi} into automata which work over arbitrary linear orders and as a consequence established a double exponential space algorithm for the satisfiability problem of this temporal logic over the class of all linear orders.

E-mail address: rabinoa@post.tau.ac.il.

0890-5401/$ – see front matter
doi:10.1016/j.ic.2011.11.002

Let *TL* be a temporal logic with a finite set of modalities definable in the existential fragment of monadic second-order logic. We prove in this paper in a uniform manner that the satisfiability problem for *TL* is in PSPACE over the following classes of time domains: (1) all linear orders, (2) ordinals, (3) scattered linear orders, (4) Dedekind-complete linear orders, (5) continuous orders, (6) rationals, (7) reals.

The proofs are based both on the composition method and on automata theoretical techniques and are easily adapted to various classes of structures and temporal and modal logics.

Recently, Reynolds [18] proved PSPACE upper bound for most of these classes, by reducing the satisfiability problem for these classes to the satisfiability problem over the reals.

Our proof uses several reductions. The first reduction uses the following notion. Let $\varphi(X_1, \ldots, X_k)$ be a formula with free set variables among $X_1, \ldots, X_k$. An instance of φ is a formula obtained by replacing $X_1, \ldots, X_k$ by monadic predicate names. Let Φ be a set of formulas. A Φ-conjunctive formula is a conjunction of instances of formulas from Φ.

Our first reduction shows that for every temporal logic $\mathcal{L}$ with a finite set of modalities definable in the existential fragment of monadic second-order logic there is a finite set Φ of first-order formulas and a linear time algorithm that reduces the satisfiability problem for $\mathcal{L}$ to the satisfiability problem for Φ-conjunctive formulas. This algorithm is based on a simple unnesting procedure and works as it is for a much broader class of modal logics.

Next, we introduce recursively definable classes of structures. Our second reduction shows that for every finite set Φ of first-order formulas and every recursively definable class of structures $\mathcal{C}$ the satisfiability problem for the Φ-conjunctive formulas over $\mathcal{C}$ is in EXPTIME. Like the first reduction, this reduction is quite general; it relies on the composition method and is sound not only for linear orders. The first two reductions give an almost free EXPTIME algorithm for many temporal and modal logics with finite sets of modalities.

To obtain a PSPACE upper bound we need more subtle arguments. We assign a rank to every structure in a recursively definable class. An algorithm similar to the algorithm in the second reduction shows that for every polynomial p the problem whether a Φ-conjunctive formula φ is satisfiable over the structures of rank $p(|\varphi|)$ is in PSPACE. The main effort to show that the satisfiability problem for a recursively definable class is in PSPACE is to establish that if a formula is satisfiable, then it is satisfiable over the structures of a polynomial rank in the size of the formula. We prove such a bound for many interesting classes of linear orders. Our proof uses an automata-theoretical characterization of the temporal logic with Stavi's modalities over the linear orders found by Cristau [6].

The paper is organized as follows. The next section recalls basic definitions about monadic second-order logic, its fragments and temporal logics. Section 3 states a linear reduction from temporal logics to conjunctive formulas. Section 4 reviews basic notions about the compositional method. Section 5 introduces recursively defined classes of structures and Section 6 presents an exponential algorithm for the satisfiability of conjunctive formulas over these classes. Section 7 presents a PSPACE algorithm for the satisfiability of conjunctive formulas over the class of all linear orders and states a small rank property lemma needed for its complexity analysis. Section 8 introduces finite base automata over arbitrary linear orders. Section 9 states the main technical lemma (Lemma 9.1) about runs of automata and proves the small rank property lemma which was used in the proof of PSPACE bound of our algorithm. Section 10 is the most technical part of the paper. It develops compositional methods for the automata types and proves the main technical lemma. Section 11 considers temporal logics with any finite set of automata definable modalities and shows that the satisfiability problem for such logics over the class of countable linear orders is in PSPACE. Section 12 proves in a "plug-and-play" manner a PSPACE upper bound over several interesting classes of linear orders and discusses related works. Section 13 discusses the related results of Mark Reynolds [16–18]. Section 14 contains conclusion and further results.

Our results were obtained in 2007 using only the composition method and the proofs were considerably simplified in July 2009, relying on the automata theoretical results of Cristau [6]. An extended abstract of this paper was published in [15].

2. Monadic logics and temporal logics

2.1. Monadic second-order logic

Monadic second-order logic (MSO) is the fragment of the full second-order logic allowing quantification only over elements and monadic predicates. One way to define the monadic second-order language for a signature Δ (notation MSO(Δ)) is to augment the first-order language for Δ by quantifiable monadic predicate variables (set variables) and by new atomic formulas $X(t)$, where t is a first-order variable and X is a monadic predicate variable. The monadic predicate variables range over all subsets of a structure for Δ.

The *quantifier depth* of a formula φ is defined as usual and is denoted by $\mathrm{qd}(\varphi)$.

We will use lower case letters t, t' for the first-order variables and upper case letters X, Y, Z for the monadic variables.

An MSO formula is existential if it is of the form $\exists X_1 \ldots \exists X_n \varphi$, where φ does not contain second-order quantifiers. The existential fragment of MSO consists of existential MSO formula and is denoted by $\exists$-MSO.

The first-order fragment of MSO contains formulas without the second-order quantifiers. These formulas might contain free second-order variables which play the same role as monadic predicate names. Hence, a formula in this fragment is interpreted over expansions of Δ-structures by predicates which provide meaning for the monadic variables. Sometimes,