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On the hardness of computing span of subcubic graphs ☆

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ABSTRACT

Given a nonempty graph G and a function ξ that assigns positive integers to the edges of G, a ξ-coloring of G is a vertex coloring of G such that for every edge uv of G the colors assigned to the vertices u and v differ by at least $\xi(uv)$. In the paper we study the problem of finding ξ-colorings with minimal span, i.e. the difference between the largest and the smallest color used. We show that the problem, restricted to subcubic graphs, is:

- NP-hard in the strong sense but polynomially $\frac{3}{2}$-approximable for functions ξ that take at most two values;
- polynomially 2-approximable and not $(1+\varepsilon)$-approximable for any $\varepsilon < \frac{1}{2}$, unless $P = NP$.

We also show that, if we additionally assume that the edges that received the largest of the values of ξ induce a spanning and connected subgraph, then it becomes

- polynomially $\frac{4}{3}$-approximable;
- polynomially solvable provided ξ takes at most two values.

1. Introduction

The frequency assignment problem (FAP), introduced by Hale in [1], can be briefly stated as follows: there are several transmitters in a certain region of a plane; assign frequencies to the transmitters in such a way that interference is avoided and the used frequency band is as small as possible. There are several graph-theoretic models for FAP but all of them share the same idea: the transmitters are represented as vertices, possible interference is modeled as edges, the frequencies are assumed to be integers and the solution to the problem is a vertex coloring satisfying some additional conditions.

The $L(p,q)$-labeling is one of these models. In the model the emphasis is placed on the interference caused by the proximity of the transmitters: the closer they are, the interference is stronger. More precisely, the $L(p,q)$-labeling problem can be stated as follows: given positive integers p, q and a graph G, find a vertex coloring of G such that the colors assigned to vertices u, v differ by at least p, if they are adjacent, and differ by at least q, if they are at distance 2. The required coloring should minimize the span, i.e. the difference between the largest and the smallest color used. See [2] for a survey of results concerning this model.

The backbone coloring problem is another model. In the problem we assume the existence of a certain substructure in a modeled network, called backbone, with higher requirements concerning the level of interference. More

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precisely, this problem can be stated as follows: given a graph G, its spanning subgraph H (the backbone) and a positive integer λ, find a vertex coloring of G such that the colors assigned to vertices adjacent in H differ by at least λ. Of course, the required coloring should minimize the span—this requirement is present in every model of FAP. See [3] for more information about this model.

In the paper we study a problem that may be viewed as a generalization of the above FAP models: given a graph G and a function ξ that assigns positive integers to the edges of G, find a vertex coloring c of G such that $|c(u) - c(v)| \geq \xi(uv)$ for every edge uv of G and the span of c is minimal. We show that this problem, restricted to subcubic graphs, is *NP*-hard in the strong sense but polynomially $\frac{3}{2}$-approximable even for functions ξ that take at most two values, polynomially 2-approximable and not $(1+\varepsilon)$-approximable for any $\varepsilon < \frac{1}{2}$, unless $P = NP$. Next, we study a subproblem in which we additionally assume that the edges that received the largest of the values of ξ induce a spanning and connected subgraph. We show that this subproblem is polynomially $\frac{4}{3}$-approximable and, if ξ takes at most two values, polynomially solvable.

Our motivation to investigate this problem for the class of subcubic graphs (i.e. graphs with degree at most 3) is twofold. First, as it is well known that all subcubic graphs except K_4 are 3-colorable, therefore the coloring problem for subcubic graphs is solvable in polynomial time, which follows from famous Brooks' theorem. Second, the following results generalize the results obtained for the backbone coloring problem in [4], where we presented an $O(n^2)$ algorithm for subcubic graphs and we proved that the problem is NP-hard for graphs with degree greater than 4.

2. Preliminaries

We begin with some definitions, notations and preliminary results. Let $G = (V, E)$ be a nonempty graph and $\xi \colon E \to \mathbb{N}$ be a function.

Definition 1. A function $c \colon V \to \mathbb{Z}$ is a ξ-*coloring* of G if and only if $|c(u) - c(v)| \geq \xi(uv)$ for all edges $uv \in E$.

Definition 2. The span of a ξ-coloring c, denoted by $\mathrm{sp}(c)$, is the difference between the largest and the smallest integer used by c.

The minimal possible span over all ξ-colorings of G, denoted by $\mathrm{sp}(G, \xi)$, will be called the ξ-*span* of G. A ξ-coloring c of G is *optimal* if and only if its span equals $\mathrm{sp}(G, \xi)$.

Proposition 1. *Let $G = (V, E)$ be a graph and $G' = (V', E')$ be a nonempty subgraph of G. If $\xi \colon E \to \mathbb{N}$ and $\xi' \colon E' \to \mathbb{N}$ are functions such that $\xi' \leq \xi|_{E'}$ then $\mathrm{sp}(G', \xi') \leq \mathrm{sp}(G, \xi)$.*

Proof. It follows immediately from the fact that if c is a ξ-coloring of G then $c|_{V'}$ is a ξ'-coloring of G'. $\square$

The greedy algorithm can be adapted to produce ξ-colorings. Given an ordering $v_1, v_2, \ldots, v_n$ of vertices of G, it assigns the first available color $c(v_i)$ to the vertex v_i for $i = 1, 2, \ldots, n$, i.e. sets $c(v_i) = 0$ if $i = 1$ or v_i has no neighbors in $v_1, v_2, \ldots, v_{i-1}$, or

$$c(v_i) = \min\{k \geq 0 \colon |k - c(v_j)| \geq \xi(v_j v_i) \text{ for all } j < i \\ \text{such that } v_j v_i \in E\},$$

otherwise.

Proposition 2. *Let $G = (V, E)$ be a nonempty graph and $\xi \colon E \to \mathbb{N}$ be a function. There is an ordering of vertices of G such that the greedy algorithm, run at this ordering, produces an optimal ξ-coloring of G.*

Proof. Let c be an optimal ξ-coloring of G such that $\min c(V) = 0$ and $v_1, v_2, \ldots, v_n$ be any ordering of vertices of G such that $c(v_1) \leq c(v_2) \leq \ldots \leq c(v_n)$. Let c' be a ξ-coloring produced by the greedy algorithm on that ordering. To complete the proof, it suffices to show that $c'(v_k) \leq c(v_k)$ for $k \leq n$. To this aim we use induction on k.

It is obvious for $k = 1$ since $c(v_1) = \min c(V) = 0$ and $c'(v_1) = 0$. Assume that $c'(v_i) \leq c(v_i)$ for $i < k$, where $2 \leq k \leq n$. Then either $c'(v_k) = 0 \leq c(v_k)$, when v_k has no previously colored neighbors, or $c'(v_k) \leq \max\{c'(v_i) + \xi(v_i v_k) \colon i < k \wedge v_i v_k \in E\} \leq \max\{c(v_i) + \xi(v_i v_k) \colon i < k \wedge v_i v_k \in E\} \leq c(v_k)$. $\square$

Corollary 3. *Let $G = (V, E)$ be a nonempty graph and $\xi \colon E \to \mathbb{N}$ be a function.*

(1) *There is an optimal ξ-coloring c of G such that for each vertex v there is a function $\zeta_v \colon E \to \{0, 1\}$ such that $c(v) = \sum_{e \in E} \xi(e)\zeta_v(e)$.*
(2) *There is a function $\zeta \colon E \to \{0, 1\}$ such that $\mathrm{sp}(G, \xi) = \sum_{e \in E} \xi(e)\zeta(e)$.*

Proof. (1) By Proposition 2 there is an ordering $v_1, v_2, \ldots, v_n$ of vertices of G such that the ξ-coloring c produced by the greedy algorithm for this ordering, is optimal. We will use induction on k to show that c has the following, stronger than required, property: there is a function $\zeta_{v_k} \colon E \to \{0, 1\}$ such that $c(v_k) = \sum_{e \in E} \xi(e)\zeta_{v_k}(e)$ and $\zeta_{v_k}(e) = 0$ if at least one of the endpoints of e has number greater than k.

It is obvious for $k = 1$ since $c(v_1) = 0$. Assume that it holds for $i < k$ ($k \geq 2$). If v_k has no neighbors in $v_1, v_2, \ldots, v_{k-1}$ or all its neighbors in $v_1, v_2, \ldots, v_{k-1}$ have been assigned colors greater than $c(v_k)$ then $c(v_k) = 0$ and clearly our claim holds. Otherwise, there is a vertex v_i such that $i < k$, $v_i v_k \in E$ and $c(v_i) < c(v_k)$. Without loss of generality we assume that $c(v_i) + \xi(v_i v_k)$ is maximal. Then $c(v_k) = c(v_i) + \xi(v_i v_k)$ since otherwise $c(v_k)$ would not be the first available color for v_k, and our claim follows immediately from the inductional assumption.

(2) It follows immediately from the proof of (1) since the coloring c used in this proof satisfies $\min c(V) = 0$. $\square$

Recall that χ usually denotes the chromatic number of G.