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On the ensemble of optimal dominating and locating-dominating codes in a graph

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ABSTRACT

Let G be a simple, undirected graph with vertex set V. For every $v \in V$, we denote by $N(v)$ the set of neighbours of v, and let $N[v] = N(v) \cup \{v\}$. A set $C \subseteq V$ is said to be a *dominating code* in G if the sets $N[v] \cap C$, $v \in V$, are all nonempty. A set $C \subseteq V$ is said to be a *locating-dominating code* in G if the sets $N[v] \cap C$, $v \in V \setminus C$, are all nonempty and distinct. The smallest size of a dominating (resp., locating-dominating) code in G is denoted by $d(G)$ (resp., $\ell(G)$).

We study the ensemble of all the different optimal dominating (resp., locating-dominating) codes C, i.e., such that $|C| = d(G)$ (resp., $|C| = \ell(G)$) in a graph G, and strongly link this problem to that of induced subgraphs of Johnson graphs.

1. Introduction

We introduce basic definitions and notation for graphs (for which we refer to, e.g., [2] and [4]), and for codes. Dominating codes constitute an old, large, classical topic (see, e.g., [5] or [6]); in the particular case when the graph is the hypercube, they are known as covering codes and have received a lot of attention in Coding Theory: see [3] and the on-line bibliography at [9], with 1000 references. Locating-dominating codes [12] are part of a larger class of codes which aim at distinguishing, in some ways, between vertices: watching systems, identifying, locating-dominating and discriminating codes, resolving sets, ...; they may have many applications and are a fast growing field, as show the 300 references in the on-line bibliography at [10], most of them published in the 21st century.

We denote by $G = (V, E)$ a simple, undirected graph with vertex set V and edge set E, where an *edge* between $x \in V$ and $y \in V$ is denoted by xy or yx. Two vertices linked by an edge are said to be *neighbours*. We denote by $N(v)$ the set of neighbours of the vertex v, and $N[v] = N(v) \cup \{v\}$. An *induced subgraph* of G is a graph with vertex set $X \subseteq V$ and edge set $\{uv \in E : u \in X, v \in X\}$. We say that two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic*, and write $G_1 \cong G_2$, if there is a bijection $\phi : V_1 \to V_2$ such that $xy \in E_1$ if, and only if, $\phi(x)\phi(y) \in E_2$ for all $x, y \in V$.

Whenever three vertices x, y, z are such that $x \in N[z]$ and $y \notin N[z]$, we say that z *separates* x and y in G (note that $z = x$ is possible). A set is said to separate x and y in G if it contains at least one vertex which does.

A *code* C is simply a subset of V, and its elements are called *codewords*. For each vertex $v \in V$, the *identifying set*

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of v, with respect to C, is denoted by $I_{G,C}(v)$ and is defined by

$$I_{G,C}(v) = N[v] \cap C.$$

We say that C is a *dominating* code in G if all the sets $I_{G,C}(v)$, $v \in V$, are nonempty.

We say that C is a *locating-dominating* code [12] if all the sets $I_{G,C}(v)$, $v \in V \setminus C$, are nonempty and distinct. In particular, any two non-codewords are separated by C. In the sequel, we shall use LD for locating-dominating.

We denote by $d(G)$ (respectively, $\ell(G)$) the smallest cardinality of a dominating (respectively, LD) code. Any dominating (respectively, LD) code C such that $|C| = d(G)$ (respectively, $|C| = \ell(G)$) is said to be *optimal*.

One application of LD codes is, for instance, fault diagnosis in multiprocessor systems: such a system can be modeled by a graph $G = (V, E)$ where V is the set of processors and E the set of links between processors. Assume that at most one processor is malfunctioning and we wish to test the system and locate the faulty processor. For this purpose, some processors (constituting the code) will be chosen and assigned the task of testing their neighbours. Whenever a selected processor (or codeword) detects a fault, it sends an alarm signal. We require that we can uniquely tell the location of the malfunctioning processor based on the information which ones of the codewords gave the alarm; under the assumption that the codewords work without failure, or that their only task is to test their neighbours (i.e., they are not considered as processors anymore) and that they perform this simple task without failure, then an LD code is what we need, because no two non-codewords have the same (nonempty) set of neighbours-codewords.

In this paper, we study the structure of the ensemble of all the optimal dominating codes and the ensemble of all the optimal LD codes of a graph. These ensembles are trivially collections of k-element subsets, or k-subsets, of V, for $k = d(G)$ or $k = \ell(G)$; we denote these ensembles by $\Xi(G)$ and $\Psi(G)$, respectively. Conversely, assume that $\mathcal{A}$ is a nonempty collection of some s different k-subsets $A_1, A_2, \dots, A_s$ of $V_1 = \{1, 2, \dots, n\}$. The question is: is there a graph G with vertex set V_1 such that $\mathcal{A}$ is equal to $\Xi(G)$ or $\Psi(G)$? When $3 \le k \le n-3$, the answer for *almost* all collections $\mathcal{A}$ is NO; indeed, there are $2^{\binom{n}{k}}$ such collections but only $2^{\binom{n}{2}}$ different graphs. However, we can ask the same question for a graph G with $n+m$ vertices, $m \ge 0$. And now the answer is YES: Theorem 2 below states that

> given any collection $\mathcal{A}$ of k-subsets of V_1, there is a positive integer m and a graph $G = (V, E)$ with $V = V_1 \cup V_2$, where $V_2 = \{n+1, \dots, n+m\}$, such that $C \subseteq V$ is an optimal dominating code in G if, and only if, $C = A$ for some $A \in \mathcal{A}$.

So the ensemble of the optimal dominating codes of the graph G can be described by which k-set of vertices from V_1 we put in the code; now these k-sets are precisely the k-sets which belong to our target $\mathcal{A}$, and therefore the set $\Xi(G)$ is equivalent to $\mathcal{A}$. If, for any two k-subsets A_i and A_j in $\mathcal{A}$ we set

$$\delta(A_i, A_j) = |A_i \Delta A_j|,$$

where Δ stands for the symmetric difference, then, setting $C_i = A_i \cup S$ and $C_j = A_j \cup S$, we can see that $\delta(C_i, C_j) = \delta(A_i, A_j)$, i.e., G is such that $\Xi(G)$ has exactly the same symmetric difference distribution as the arbitrary collection $\mathcal{A}$ we started from.

Theorem 3 gives a similar result for LD codes, with similar consequences for $\Psi(G)$; also, the same kind of result is proved for identifying codes, which we do not define here, in [7].

Now, this establishes a sufficiently strong link, between the ensembles of the optimal dominating or LD codes of all graphs and the sets of k-subsets of n-sets, to connect our investigation to the following definition from [11] and the results related to it; see also [1].

Definition 1. Given positive integers k and n with $1 \le k \le n$, the Johnson graph $J(k, n)$ is the graph whose vertex set consists of all the k-subsets of $\{1, 2, \dots, n\}$, with edges between two vertices sharing exactly $k-1$ elements.

A graph H is isomorphic to an induced subgraph of a Johnson graph if, and only if, it is possible to assign, for some k and n, a k-subset $S_v \subseteq \{1, 2, \dots, n\}$ to each vertex v of H in such a way that distinct vertices have distinct corresponding k-sets, and vertices v and w are neighbours if, and only if, S_v and S_w share exactly $k-1$ elements. In this case, we say that H is an induced subgraph of a Johnson graph, or that H is a JIS for short.

We denote by $\mathcal{J}$ the set of all induced subgraphs of all Johnson graphs.

If we link two elements C_i and C_j in $\Xi(G)$ (respectively, $\Psi(G)$) if, and only if, $\delta(C_i, C_j) = 2$, then we obtain a graph which we denote by $\mathcal{N}(G)$ (respectively, $\mathcal{M}(G)$), and the set of all the graphs $\mathcal{N}(G)$ (respectively, $\mathcal{M}(G)$) is denoted by $\mathcal{N}$ (respectively, $\mathcal{M}$). Now, what Theorems 2 and 3 show as an immediate consequence is that

> every JIS belongs to $\mathcal{N}$, or: $\mathcal{J} = \mathcal{N}$;
> every JIS belongs to $\mathcal{M}$, or: $\mathcal{J} = \mathcal{M}$.

For examples of graphs which are JIS or not, we refer to [11], with a short overview in Section 3, but to our knowledge no classification is known.

2. Main results

Theorem 2. *Let $1 \le k \le n$ be an arbitrary integer, and assume that $\mathcal{A}$ is any nonempty collection of k-subsets of $V_1 = \{1, 2, \dots, n\}$. Then there is a positive integer m and a graph G with vertex set $V = V_1 \cup V_2$, where $V_2 = \{n+1, n+2, \dots, n+m\}$, such that $C \subseteq V$ is an optimal dominating code in G if, and only if, $C = A$ for some $A \in \mathcal{A}$.*

Proof. Denote by $\mathcal{B}$ the set of all $(k-1)$-subsets of V_1 together with all the k-subsets of V_1 that do not belong to $\mathcal{A}$; this set has size $\binom{n}{k-1} + \binom{n}{k} - |\mathcal{A}|$.

We begin the construction of G by taking n vertices $a_1, a_2, \dots, a_n$, which we link together in all possible ways, so as to form the clique K_n (these vertices play the role of