Information Processing Letters 115 (2015) 707–711

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

# Perfect codes in direct graph bundles

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ARTICLE INFO

*Article history:*
Received 10 November 2014
Received in revised form 20 March 2015
Accepted 20 March 2015
Available online 26 March 2015
Communicated by Ł. Kowalik

*Keywords:*
Combinatorial problems
Cycles
Direct graph bundle
Perfect codes

ABSTRACT

A complete characterization of perfect codes in direct graph bundles of cycles over cycles is given.



## 1. Introduction

The study of codes in graphs presents a wide generalization of the problem of the existence of (classical) error-correcting codes. In general, for a given graph $G$ we search for a subset $X$ of its vertices such that the $r$-balls centered at vertices from $X$ form a partition of the vertex set of $G$. Hamming codes and Lee codes correspond to codes in the cartesian product of complete graphs and cycles, respectively.

The study of codes in graphs was initiated by Biggs [1], who rightly noticed that the class of all graphs is too general a setting, and hence restricted himself to distance-transitive graphs. Kratochvíl continued the study of (perfect) codes in graphs, for instance, in [11] he proved the remarkable result that there are no nontrivial 1-perfect codes over complete bipartite graphs with at least three vertices. (Here "over" means with respect to the cartesian product powers of such graphs.)

Besides being of interest in complexity theory [2], the problem is of obvious practical interest, therefore perfect codes on various classes of graphs are extensively studied. Interesting graph classes include graph products and bundles, which appear to be among widely used topologies for computer systems architecture, cf. the famous ILLIAC IV, however sometimes under different names (see [3]). The concept has applications in game theory and frequency assignment [11,4]. Perfect codes of direct product graphs and in particular, perfect codes in products of cycles were studied recently [7–10,14]. In this paper, we study existence of perfect codes in direct graph bundles of cycles over cycles and provide a complete solution.

The rest of the paper is organized as follows. In the next section we provide basic terminology and notation, and continue with some basic facts in Section 3. In Sections 4, 5 and 6 we prove the propositions that are summarized in the following two theorems.

**Theorem 1.1.** *Let $r \geq 1$, $m, n \geq 3$, and $t = (r+1)^2 + r^2$. Let $X = C_m \times^{\sigma_\ell} C_n$ be a direct graph bundle with fiber $C_n$ and base $C_m$. Then each connected component of $X$ contains an $r$-perfect code if and only if $n$ is a multiple of $t$, $m > r$, and $\ell$ has a form of $\ell = (\alpha t \pm ms) \bmod n$ for some $\alpha \in \mathbb{Z}$.*

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1 Supported in part by ARRS, the Research Agency of Slovenia.

http://dx.doi.org/10.1016/j.ipl.2015.03.010

**Theorem 1.2.** *There is no r-perfect code of (a connected component of) direct graph bundle $C_m \times^\alpha C_n$ where $\alpha$ is a reflection.*

## 2. Terminology and notation

A finite, simple and undirected graph $G = (V(G), E(G))$ is given by a set of vertices $V(G)$ and a set of edges $E(G)$. As usual, the edge $\{i, j\} \in E(G)$ is shortly denoted by $ij$. Two graphs $G$ and $H$ are called *isomorphic*, in symbols $G \simeq H$, if there exists a bijection $\varphi$ from $V(G)$ onto $V(H)$ that preserves adjacency and nonadjacency. An isomorphism of a graph $G$ onto itself is called an *automorphism*. The identity automorphism on $G$ will be denoted by $id_G$ or shortly $id$. The *cycle* $C_n$ on $n$ vertices is defined by $V(C_n) = \{0, 1, \ldots, n-1\}$ and $ij \in E(C_n)$ if $i = (j \pm 1) \bmod n$. Denote by $P_n$ the *path* on $n \geq 1$ distinct vertices $0, 1, 2, \ldots, n-1$ with edges $ij \in E(P_n)$ if $j = i+1$, $0 \leq i < n-1$.

Let $G$ and $H$ be connected graphs. The *direct product* of graphs $G$ and $H$ is the graph $G \times H$ with vertex set $V(G \times H) = V(G) \times V(H)$ and whose edges are all pairs $(g_1, h_1)(g_2, h_2)$ with $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. The direct product of graphs is commutative and associative in a natural way. For more facts on the direct product of graphs and other graph products we refer to [5].

Graph bundles are a natural generalization of graph products [12]. Let $B$ and $G$ be graphs and $\mathrm{Aut}(G)$ be the set of automorphisms of $G$. To any ordered pair of adjacent vertices $u, v \in V(B)$ we assign an automorphism of $G$. Formally, let $\alpha : V(B) \times V(B) \to \mathrm{Aut}(G)$. For brevity, we will write $\alpha(u, v) = \alpha_{u,v}$ and assume that $\alpha_{v,u} = \alpha_{u,v}^{-1}$ for any $u, v \in V(B)$. We construct the graph $X$ as follows. The vertex set of $X$ is the Cartesian product of vertex sets, $V(X) = V(B) \times V(G)$. The edges of $X$ are given by the rule: for any $b_1 b_2 \in E(B)$ and any $g_1 g_2 \in E(G)$, the vertices $(b_1, g_1)$ and $(b_2, \alpha_{b_1,b_2}(g_2))$ are adjacent in $X$. We call $X$ a *direct graph bundle* with base $B$ and fibre $G$ and write $X = B \times^\alpha G$.

Clearly, if all $\alpha_{u,v}$ are identity automorphisms, the graph bundle is isomorphic to the direct product $X = B \times^\alpha G = B \times G$. Furthermore, it is well-known that if the base graph is a tree, then the graph bundle is always isomorphic to a product, i.e. $X = T \times^\alpha G \simeq T \times G$ for any graph $G$, any tree $T$ and any assignment of automorphisms $\alpha$ [12].

A graph bundle over a cycle can always be constructed in a way that all but at most one automorphism are identities. Fixing $V(C_n) = \{0, 1, 2, \ldots, n-1\}$, let us denote $\alpha_{n-1,0} = \alpha$, $\alpha_{i-1,i} = id$ for $i = 1, 2, \ldots, n-1$, and write $C_n \times^\alpha G$.

A graph bundle $C_n \times^\alpha G$ can be represented also as the graph obtained from the product $P_n \times G$ by adding (edges of) a copy of $K_2 \times G$ between vertex sets $\{n-1\} \times V(G)$ and $\{0\} \times V(G)$ such that if $V(K_2) = \{1, 2\}$ and $(1, u)$ is adjacent to $(2, v)$ in $K_2 \times G$, then $(n-1, u)$ and $(0, \alpha(v))$ are connected by an edge in $C_n \times^\alpha G$. The natural projection $p : C_n \times^\alpha G \to C_n$ is called the *bundle projection*. The preimage $p^{-1}(u)$ is called the *fiber* over $u$, denoted $F_u$.

Automorphisms of a cycle are of two types. A cyclic shift of the cycle by $\ell$ elements, denoted by $\sigma_\ell$, $0 \leq \ell < n$, maps $u_i$ to $u_{i+\ell}$ (indices are modulo $n$). As a special case we have the identity ($\ell = 0$). Other automorphisms of cycles are reflections. Depending on parity of $n$ the reflection of a cycle may have one, two or no fixed points. More formally, we define:

- cyclic $\ell$-shift, $\sigma_\ell$ defined as $\sigma_\ell(i) = i + \ell$ for $i = 0, 1, \ldots, n-1$.
- reflection with no fixed points $\rho_0$ defined as $\rho_0(i) = n - i - 1$ for $i = 0, 1, \ldots, n-1$. ($n$ even.)
- reflection with one fixed point $\rho_1$ defined as $\rho_1(i) = n - i - 1$ for $i = 0, 1, \ldots, n-1$. ($n$ odd; there is exactly one fixed point, $\rho_1(\frac{n-1}{2}) = n - \frac{n-1}{2} - 1 = \frac{n-1}{2}$.)
- reflection with two fixed points $\rho_2$ defined as $\rho_2(0) = 0$ and $\rho_2(i) = n - i$ for $i = 1, 2, \ldots, n-1$. ($n$ even, the second fixed point is $\rho_2(\frac{n}{2}) = n - \frac{n}{2} = \frac{n}{2}$.)

Note that in the definition of shifts, the summation is calculated modulo $n$. Throughout the paper, the summations in first coordinates of vertices will be calculated modulo $m$ and in second coordinates modulo $n$.

For a graph $G = (V, E)$ the *distance* $d_G(u, v)$, or briefly $d(u, v)$, between vertices $u$ and $v$ is defined as the number of edges on a shortest walk from $u$ to $v$. A *walk* is a sequence $v_0, e_0, v_1, e_1, v_2, \ldots, v_k$ of graph vertices $v_i$ and graph edges $e_i$ such that for $1 \leq i \leq k$, the edge $e_i$ has endpoints $v_{i-1}$ and $v_i$. For a vertex $v \in V$ let $B_r(v) = \{u \in V \mid d(u, v) \leq r\}$ be the $r$-ball centered at $v$. In particular, $N[v] = B_1(v)$ and $N(v) = N[v] \setminus \{v\}$. A set $C \subseteq V$ is an *r-code* in $G$ if $B_r(u) \cap B_r(v) = \emptyset$ for any two distinct vertices $u, v \in C$. In addition, an $r$-code $C$ is called an *r-perfect code* if $\{B_r(u) \mid u \in C\}$ forms a partition of $V$.

## 3. Preliminaries

Let us denote $V(P_m) = V_0(m) \cup V_1(m)$ where $V_0(m) = \{0, 2, 4, \ldots, 2\lfloor \frac{m-1}{2} \rfloor\}$ and $V_1(m) = \{1, 3, \ldots, 2\lceil \frac{m-1}{2} \rceil - 1\}$ are the sets of *even* and *odd* vertices.

Let $Z_0(m, n) = (V_0(m) \times V_0(n)) \cup (V_1(m) \times V_1(n))$ denote the *even component* of $P_m \times P_n$ and $Z_1(m, n) = (V_0(m) \times V_1(n)) \cup (V_1(m) \times V_0(n))$ denote the *odd component* of $P_m \times P_n$. The terms even component (odd component) have been chosen because vertices $(i, j)$ in even (odd) component are exactly those for which $i + j$ is even (odd). It is not difficult to see that $Z_0(m, n)$ consists of $\lceil \frac{mn}{2} \rceil$ vertices and $(m-1)(n-1)$ edges [5].

Jha [8] showed that for $r \geq 1$ and $m, n \geq 2r+2$ an $r$-ball in $C_m \times C_n$ is isomorphic to $Z_0(2r+1, 2r+1)$. Delete all connections between the fiber $F_{m-1}$ and $F_0$ in $C_m \times^\alpha C_n$ to obtain a graph $P_m \times C_n$. Fibers $F_{m-1}$ and $F_0$ and deleted connections are a subgraph of $C_m \times^\alpha C_n$ isomorphic to $P_2 \times C_n$, so we can conclude:

**Lemma 3.1.** *Let $r \geq 1$, $m, n \geq 2r+1$ and let $\alpha$ be an automorphism of $C_n$. Then an r-ball in $C_m \times^\alpha C_n$ is isomorphic to $Z_0(2r+1, 2r+1)$.*

Two examples of an 2-ball in $C_m \times^\alpha C_n$ are given in Fig. 1.

The fact that the direct product $F \times G$ of connected and bipartite factors $F$ and $G$ has exactly two components was first proved by Weichsel [13]. In particular, $C_m \times C_n$,