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Notes on vertex pancyclicity of graphs ☆

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ABSTRACT

In (2012) [7], Kewen Zhao and Yue Lin introduced a new sufficient condition for pancyclic graphs and proved that if G is a 2-connected graph of order $n \geqslant 6$ with $|N(x) \cup N(y)| + d(w) \geqslant n$ for any three vertices x, y, w of $d(x, y) = 2$ and wx or $wy \notin E(G)$, then G is 4-vertex pancyclic or G belongs to two classes of well-structured exceptional graphs. This result generalized the two results of Bondy in 1971 and Xu in 2001. In this paper, we first prove that if G is a 2-connected graph of order $n \geqslant 6$ with $|N(x) \cup N(y)| + d(w) \geqslant n$ for any three vertices x, y, w of $d(x, y) = 2$ and wx or $wy \notin E(G)$, then each vertex u of G with $d(u) \geqslant 3$ is 5-pancyclic or $G = K_{n/2,n/2}$, and we also show that our result is best possible. On the basis of this result, we prove that there exist at least two pancyclic vertices in G or $G = K_{n/2,n/2}$. In addition, we give a new proof of a result in Cai (1984) [2].

1. Introduction

In this paper, we consider only finite undirected graphs with no loops or multiples. Let $G = (V(G), E(G))$ be a graph. We denote the number of vertices of G by $|V(G)|$. A subdigraph induced by a subset $X \subseteq V(G)$ is denoted by $G[X]$. We also write $G - X$ for $G[V(G) - X]$.

For a vertex x and a subgraph H of G, the set of all vertices being adjacent to x in H is denoted by $N_H(x)$. Furthermore, $d_H(x) = |N_H(x)|$ is *degree* of x in H. We use $\delta(G) = \min\{d_G(x)\colon\ x \in V(G)\}$ to stand for the *minimum degree* of G. When there is no confusion possible, we use $N(x)$, $d(x)$ and δ instead of $N_G(x)$, $d_G(x)$ and $\delta(G)$, respectively. Let R, H be two subgraphs of G. We use $N_H(R)$ to denote the set of all vertices in H being adjacent to some vertex in R.

An l-cycle is a cycle of length l. We call a graph G Hamiltonian if it contains a cycle of length $|V(G)|$. A graph G is called to be pancyclic if G contains cycles of length from 3 to $|V(G)|$. A vertex is said to be *k-pancyclic* in a graph G, if it belongs to an l-cycle for all $k \leqslant l \leqslant |V(G)|$. For $k = 3$, we also say that the vertex is *pancyclic*.

In 1960, Ore [5] introduced degree sum condition for a graph G to be Hamiltonian. In 1971, Bondy considered the above Ore's condition for pancyclic graphs and proved that

Theorem 1.1. *(See Bondy [1].) If G is a 2-connected graph of order n with $d(x) + d(y) \geqslant n$ for each pair of non-adjacent vertices x, y in G, then G is pancyclic or $G = K_{n/2,n/2}$.*

In 1984, Xiaotao Cai considered the above Ore's condition for vertex-pancyclic graphs and proved that

Theorem 1.2. *(See Cai [2].) If G is 2-connected graph of order $n \geqslant 4$ with $d(x) + d(y) \geqslant n$ for each pair of non-adjacent vertices x, y in G, then G is 4-vertex pancyclic or $G = K_{n/2,n/2}$.*

In 1989, Lindquester [4] introduced the condition on neighborhood union of each pair vertices at distance 2 for a graph G to be Hamiltonian. In 2001, Xu generalized Lindquester's result and proved the following pancyclic result.

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Theorem 1.3. *(See Xu [6].) If G is a 2-connected graph of order $n \geqslant 6$ with $|N(x) \cup N(y)| + \delta \geqslant n$ for each pair of non-adjacent vertices x, y of $d(x, y) = 2$ in G, then G is pancyclic or $G = K_{n/2,n/2}$.*

Lin and Song [3] improved Theorem 1.3 and proved that

Theorem 1.4. *(See Lin and Song [3].) If G is a 2-connected graph of order $n \geqslant 6$ with $|N(x) \cup N(y)| + \delta \geqslant n$ for each pair of non-adjacent vertices x, y of $d(x, y) = 2$ in G, then G is 4-vertex pancyclic or $G = K_{n/2,n/2}$.*

Recently, Kewen Zhao and Yue Lin presented a new sufficient condition and proved the following Theorem 1.5 which generalized Theorem 1.1 and Theorem 1.3.

Theorem 1.5. *(See Zhao and Lin [7].) If G is a 2-connected graph of order $n \geqslant 6$ with $|N(x) \cup N(y)| + d(w) \geqslant n$ for any three vertices x, y, w of $d(x, y) = 2$ and wx or $wy \notin E(G)$, then G is 4-vertex pancyclic or G belongs to two classes of well-structured exceptional graphs.*

In this paper, let G be as in Theorem 1.5, we first prove that each vertex u of G with $d(u) \geqslant 3$ is 5-pancyclic or $G = K_{n/2,n/2}$. We also show that our result is best possible. On the basis of this result, we prove that there exist at least two pancyclic vertices in G or $G = K_{n/2,n/2}$, and we also provide a new proof of Theorem 1.2.

2. Main results

Below, we use C_l to stand for an l-cycle for any integer l.

Lemma 2.1. *Let G be as in Theorem 1.5 and $G \neq K_{n/2,n/2}$. Then the following hold.*

(1) *G contains vertices with degree greater than 2.*
(2) *G has a 3-cycle.*
(3) *Let u be a vertex of G with $d(u) \geqslant 3$. If u is in a C_3, then u is in C_3, C_4 or C_5, C_6. If u is not in a C_3, then u is in C_4, C_5.*

Proof. (1) If $d(u) = 2$ for each $u \in V(G)$, then G is a cycle $C_n = x_1x_2 \ldots x_nx_1$. Since $n \geqslant 6$, we have $|N(x_1) \cup N(x_3)| + d(x_5) \leqslant n - 1$, a contradiction. Therefore, G contains vertices with degree greater than 2.

Below, we prove (2) and (3). Let u be a vertex in G with $d(u) \geqslant 3$.

Case 1: u is in a C_3.

In this case, G has a 3-cycle. Let $C_3 = uvwu$ be a 3-cycle containing the vertex u. Since $n \geqslant 6$ and G is 2-connected, we have that the degree of each of u, v, w is not less than 2 and there exist at least two vertices in u, v, w with degree greater than 2. Note that $d(u) \geqslant 3$. Without loss of generality, we assume that $d(v) \geqslant 3$, $d(w) \geqslant 2$. Let $B = V(G - N(u) - \{u\})$.

Suppose that u is not a 4-cycle in G. Clearly, we have $G[N(u)]$ has not path of length 2 and each pair of vertices in $N(u)$ has not common neighbor in $G - \{u\}$. By $d(v) > 2$, we have $N_B(v) \neq \emptyset$. Let $v_1 \in N_B(v)$. Since G is 2-connected and $d(u) \geqslant 3$, there exists at least one vertex $z \in N(u)\backslash\{v, w\}$ such that $N_B(z) \neq \emptyset$. Let $z_1 \in N_B(z)$.

Subcase 1.1: $d(w) = 2$.

If there is a vertex in B which is not adjacent to v_1, then $|N(u) \cup N(v_1)| + d(w) \leqslant d(u) + (|B| - 2) + 2 < n$, a contradiction. So each of vertices in $B - \{v_1\}$ is adjacent to v_1. Now, we have $v_1z_1 \in E(G)$ and $d(v_1, z) = 2$. Since u is not in a 4-cycle, we have $|N(u) \cap N(z)| \leqslant 1$. When $|N(u) \cap N(z)| = 0$, we have $|N(z) \cup N(v_1)| + d(w) \leqslant (|B| + 1) + 2 = |B| + 3 \leqslant |B| + |N(u)| < n$, a contradiction. When $|N(u) \cap N(z)| = 1$, let $y \in N(u) \cap N(z)$. It is easy to see that $y \notin \{v, w\}$ and $|N(u)| \geqslant 4$. Now, we have $|N(z) \cup N(v_1)| + d(w) \leqslant (|B| + 2) + 2 = |B| + 4 \leqslant |B| + |N(u)| < n$, a contradiction. Therefore, in this case, we have that u is in a C_4.

Subcase 1.2: $d(w) > 2$.

In this case, we will prove that u is in C_5 and C_6. If z_1 and each of $\{v, w\}$ have not common neighbor, then $|N(u) \cup N(z_1)| + d(v) \leqslant n - (d(v) - 2) - (d(w) - 2) - |\{u, z_1\}| + d(v) = n - d(w) + 2 < n$, a contradiction. Therefore, there exist two vertices, say v_1 and w_1, such that $v_1 \in N_B(v)$ and $w_1 \in N_B(w)$ and z_1 is adjacent to v_1 or w_1. Now, uvv_1z_1zu or uww_1z_1zu is a C_5 and $uwvv_1z_1zu$ or $uvww_1z_1zu$ is a C_6 containing the vertex u.

Case 2: u is not in any C_3.

If u is not in a C_3, then $N(u)$ has not two adjacent vertices. We will prove that G has a 3-cycle and u is contained in C_4, C_5. Let $B = V[G - N(u) - \{u\}]$ and $v, w, z \in N(u)$.

It is easy to see that there exist two vertices in $N(u)$ that have a common neighbor in B. Otherwise, we have $|N(v) \cup N(w)| + d(z) = |N(v) \cup N(w)| + |N(z)| = |N(v) \cup N(w) \cup N(z)| + |\{u\}| \leqslant |B| + |\{u\}| + |\{u\}| = |B| + 2 < n$, a contradiction. Assume without loss of generality that w, z are adjacent to a vertex y in B. Now, $uwyzu$ is a C_4 containing the vertex u.

Below, we only need to prove that G has a 3-cycle and u is contained in a C_5.

By $d(w, z) = 2$ and $2|B| + 2 \geqslant |N(w) \cup N(z)| + d(v) \geqslant n = |B| + d(u) + 1$, we can get $|B| \geqslant d(u) - 1$.

Subcase 2.1: $|B| = d(u) - 1$.

In this case, we claim that each vertex of $N(u)$ must be adjacent to each vertex of B. (Otherwise, there is a vertex, say v, that is not adjacent to some vertex in B, then we can check that $|N(w) \cup N(z)| + d(v) \leqslant |B| + 1 + ((|B| - 1) + 1) = 2|B| + 1 = |B| + (d(u) - 1) + 1 < n$, a contradiction.) If any two vertices in B are not adjacent, then $G = K_{n/2,n/2}$, a contradiction. So there exist two adjacent vertices x, y in B, and clearly, $vxyv$ is a 3-cycle of G and $uvxywu$ is a C_5 containing the vertex u.

Subcase 2.2: $|B| \geqslant d(u)$.

If $B - N_B(N(u)) \neq \emptyset$, then there must exist a vertex in $B - N_B(N(u))$, say x, and a vertex in $N(u)$, say v, such that $d(v, x) = 2$. However, we have $|N(v) \cup N(x)| + d(u) \leqslant |B| - 1 + 1 + d(u) < n$, a contradiction.

Suppose now that $B = N_B(N(u))$. We claim that for any vertex in $N(u)$, there exist at least two vertices in B that are adjacent to it. (Otherwise, assume that there is a vertex $v \in N(u)$, such that there exists at most one vertex in B