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Edge-fault tolerance of hypercube-like networks ☆

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ABSTRACT

This paper considers a kind of generalized measure $\lambda_s^{(h)}$ of fault tolerance in a hypercube-like graph G_n which contains several well-known interconnection networks such as hypercubes, varietal hypercubes, twisted cubes, crossed cubes, Möbius cubes and the recursive circulant $G(2^n, 4)$, and proves $\lambda_s^{(h)}(G_n) = 2^h(n-h)$ for any h with $0 \leqslant h \leqslant n-1$ by the induction on n and a new technique. This result shows that at least $2^h(n-h)$ edges of G_n have to be removed to get a disconnected graph that contains no vertices of degree less than h. Compared with previous results, this result enhances fault-tolerant ability of the above-mentioned networks theoretically.

1. Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network. For graph terminology and notation not defined here we follow [20].

The edge-connectivity of a graph G is an important measurement for fault tolerance of the network, and the larger the edge-connectivity is, the more reliable the network is. However, computing this parameter, one implicitly assumes that all links incident with the same processor may fail simultaneously. Consequently, this measurement is inaccurate for large-scale processing systems in which some subsets of system components cannot fail at the same time in real applications. To overcome such a shortcoming, Esfahanian [7] proposed the concept of restricted connectivity, in which the links incident with the same processor cannot fail at the same time. Latifi et al. [11] generalized it to the restricted h-connectivity, in which at least h links incident with the same processor cannot fail. This parameter can measure fault tolerance of an interconnection network more accurately than the classical connectivity. The concepts stated here are slightly different from theirs.

For a given integer h ($\geqslant 0$), an edge subset F of a connected graph G is called an *h-super edge-cut*, or *h-edge-cut* for short, if $G - F$ is disconnected and has the minimum degree $\delta(G - F) \geqslant h$. The *h-super edge-connectivity* of G, denoted by $\lambda_s^{(h)}(G)$, is defined as the minimum cardinality over all h-edge-cuts of G. It is clear that $\lambda_s^{(0)}(G) = \lambda(G)$, where $\lambda(G)$ is classical edge-connectivity of G. For $h \geqslant 1$, if $\lambda_s^{(h)}(G)$ exists, then $\lambda_s^{(h-1)}(G) \leqslant \lambda_s^{(h)}(G)$.

For any graph G and a given integer h, determining $\lambda_s^{(h)}(G)$ is quite difficult since Latifi et al. [11] conjectured it is NP-hard, not proved so far. In fact, the existence of $\lambda_s^{(h)}(G)$ is an open problem so far when $h \geqslant 1$. Only few results have been known on $\lambda_s^{(h)}(G)$ for particular classes of graphs and small h's, such as, Xu [19] determined $\lambda_s^{(h)}(Q_n) = 2^h(n-h)$ for $h \leqslant n-1$.

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It is widely known that the hypercube has been one of the most popular interconnection networks for parallel computer/communication system. However, the hypercube has the large diameter correspondingly. To minimize diameter, various networks are proposed by twisting some pairs of links in hypercubes, such as the varietal hypercube VQ_n [5], the twisted cube TQ_n [1,2], the locally twisted cube LTQ_n [21], the crossed cube CQ_n [8,10], the Möbius cube MQ_n [6], the recursive circulant $G(2^n, 4)$ [13] and so on. Because of the lack of the unified perspective on these variants, results of one topology are hard to be extended to others. To make a unified study of these variants, Vaidya et al. [16] introduced the class of hypercube-like graphs HL_n, which contains all the above-mentioned networks. Thus, the hypercube-like graphs have received much attention in recent years [3,4,12,14,15,17,18].

In this paper, we determine $\lambda_s^{(h)}(G_n) = 2^h(n-h)$ for any $G_n \in HL_n$ and $0 \leqslant h \leqslant n-1$. Our result contains many know conclusions and enhances the fault-tolerant ability of the hypercube-like networks theoretically.

The proof of this result is in Section 3 by the induction on n and a new technique. Section 2 recalls the definition and Section 4 gives a conclusion on our work.

2. Hypercube-like graphs

Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two disjoint graphs with the same order, σ a bijection from V_0 to V_1. A 1-1 connection between G_0 and G_1 is defined as an edge-set $M_\sigma = \{x\sigma(x) \mid x \in V_0,\ \sigma(x) \in V_1\}$. Let $G_0 \oplus_\sigma G_1$ denote a graph $G = (V_0 \cup V_1, E_0 \cup E_1 \cup M_\sigma)$. Clearly, M_σ is a perfect matching of G. Moreover, if σ is the identical permutation on $V(G_0)$, then $G_0 \oplus_\sigma G_0 = G_0 \times K_2$, where $\times$ denotes the Cartesian product, and K_2 is a complete graph of order two.

Note that the operation $\oplus_\sigma$ may generate different graphs according to different σ. Applying the operation $\oplus_\sigma$ repeatedly, a set of n-dimensional *hypercube-like graphs* also called *bijective connection graphs* (in brief, *BC graphs*) [15], denoted by HL_n, can be recursively defined as follows.

(1) $HL_0 = \{G_0\}$, where $G_0 = K_1$, which is a single vertex;
(2) $G_n \in HL_n$ if and only if $G_n = G_{n-1} \oplus_\sigma G'_{n-1}$ for some $G_{n-1}, G'_{n-1} \in HL_{n-1}$, where σ is a bijection from $V(G_{n-1})$ to $V(G'_{n-1})$.

It is clear that for a graph $G_n \in HL_n$, G_n is an n-regular connected graph of order 2^n and $\lambda(G_n) = n$ (see [16]). A hypercube-like graph in HL_4 is shown in Fig. 1, which is isomorphic to $G(2^4, 4)$.

By definitions, it is easy to see that the hypercube $Q_n = Q_{n-1} \oplus_{\sigma_1} Q_{n-1}$, the varietal hypercube $VQ_n = VQ_{n-1} \oplus_{\sigma_2} VQ_{n-1}$, the twisted cube $TQ_n = TQ_{n-1} \oplus_{\sigma_3} TQ_{n-1}$, the locally twisted cube $LTQ_n = LTQ_{n-1} \oplus_{\sigma_4} LTQ_{n-1}$, the crossed cube $CQ_n = CQ_{n-1} \oplus_{\sigma_5} CQ_{n-1}$, the Möbius cube $MQ_n = MQ_{n-1} \oplus_{\sigma_6} MQ_{n-1}$, where σ_i is a given permutation on the vertex-set of the corresponding G_{n-1} in HL_{n-1} for each $i \in \{1, 2, \ldots, 6\}$. As regards the recursive circulant $G(2^n, 4)$, when $n \in \{2, 3, 4\}$, there is a permutation σ on $V(G(2^{n-1}, 4))$ such that $G(2^n, 4) = G(2^{n-1}, 4) \oplus_\sigma G(2^{n-1}, 4)$ (see Fig. 1). In general, $G(2^n, 4)$ cannot be obtained from the operation $\oplus_\sigma$ on two recursive circulants. In other words, for an arbitrary $G(2^n, 4)$, there is no a permutation σ on $V(G(2^{n-1}, 4))$ such that $G(2^n, 4) = G(2^{n-1}, 4) \oplus_\sigma G(2^{n-1}, 4)$. However, Kim et al. [9] pointed out that there is a permutation σ on $V(G(2^{n-2} \times K_2, 4))$ such that $G(2^n, 4) = [G(2^{n-2}, 4) \times K_2] \oplus_\sigma [G(2^{n-2}, 4) \times K_2]$. Thus, $\{Q_n, VQ_n, TQ_n, LTQ_n, CQ_n, MQ_n, G(2^n, 4)\} \subseteq HL_n$.

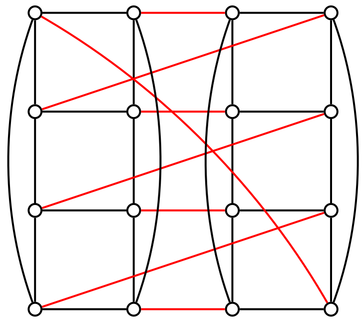

Fig. 1. A hypercube-like graph in HL_4.

For convenience, let $I_n = \{0, 1, \ldots, n\}$. For a graph G, we write $|G|$ for $|V(G)|$, for a subgraph $X \subseteq G$, write X for $V(X)$. For each $i \in I_{n-1}$, if $G_i, G'_i \in HL_i$ and $G_{i+1} = (V(G_i) \cup V(G'_i), E(G_i) \cup E(G'_i) \cup M_{\sigma_i})$, we write M_i for M_{σ_i}, and say that G_i and G'_i are the i-dimensional underlying graphs of G_{i+1} with σ_i.

Lemma 2.1. *For given $h \in I_{n-1}$ and $G_n \in HL_n$, there is a sequence of graphs $\{G_h, G_{h+1}, \ldots, G_{n-1}, G_n\}$ such that G_i is one of the i-dimensional underlying graphs of G_{i+1} for each i with $h \leqslant i \leqslant n-1$.*

Proof. From the recursive definition of HL_n, for the given graph $G_n \in HL_n$, there are two $(n-1)$-dimensional underlying graphs $G_{n-1}, G'_{n-1} \in HL_{n-1}$ with σ_{n-1} such that $G_n = G_{n-1} \oplus_{\sigma_{n-1}} G'_{n-1}$; for the graph $G_{n-1} \in HL_{n-1}$ there are two $(n-2)$-dimensional underlying graphs $G_{n-2}, G'_{n-2} \in HL_{n-2}$ with σ_{n-2} such that $G_{n-1} = G_{n-2} \oplus_{\sigma_{n-2}} G'_{n-2}$. In general, for each i with $h \leqslant i \leqslant n-1$ and the graph $G_{i+1} \in HL_{i+1}$, there are two i-dimensional underlying graphs $G_i, G'_i \in HL_i$ with σ_i such that $G_{i+1} = G_i \oplus_{\sigma_i} G'_i$. Thus the lemma follows. □

3. Main results

In this section, our aim is to prove that $\lambda_s^{(h)}(G_n) = 2^h(n-h)$ for any $G_n \in HL_n$ and $h \in I_{n-1}$.

Lemma 3.1. $\lambda_s^{(h)}(G_n) \leqslant 2^h(n-h)$ *for any $G_n \in HL_n$ and $h \in I_{n-1}$.*

Proof. Let $G_n \in HL_n$. By Lemma 2.1 there is a sequence of graphs $\{G_h, G_{h+1}, \ldots, G_{n-1}, G_n\}$ such that G_i is one of the i-dimensional underlying graphs of G_{i+1} for each i with $h \leqslant i \leqslant n-1$. Let F be the set of edges between G_h and $G_n - G_h$. Then F is an edge-cut of G_n. Since G_n is n-regular and G_h is h-regular, $|F| = |G_h|(n-h) = 2^h(n-h)$.

We now show that F is an h-edge-cut of G_n by proving $\delta(G_n - F) \geqslant h$. Let x be a vertex in $G_n - F$. If x is in G_h,