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The complexity of the proper orientation number

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ABSTRACT

A proper orientation of a graph $G=(V,E)$ is an orientation D of $E(G)$ such that for every two adjacent vertices v and u, $d^-_D(v) \neq d^-_D(u)$ where $d^-_D(v)$ is the number of edges with head v in D. The proper orientation number of G is defined as $\overrightarrow{\chi}(G) = \min_{D\in\Gamma} \max_{v\in V(G)} d^-_D(v)$ where Γ is the set of proper orientations of G. We have $\chi(G)-1 \leqslant \overrightarrow{\chi}(G) \leqslant \Delta(G)$, where $\chi(G)$ and $\Delta(G)$ denote the chromatic number and the maximum degree of G, respectively. We show that, it is **NP**-complete to decide whether $\overrightarrow{\chi}(G)=2$, for a given planar graph G. Also, we prove that there is a polynomial time algorithm for determining the proper orientation number of 3-regular graphs. In sharp contrast, we will prove that this problem is **NP**-hard for 4-regular graphs.

1. Introduction

Graph orientation is a well-studied area of graph theory, that provides a connection between directed and undirected graphs [13]. There are several problems concerned with orienting the edges of an undirected graph in order to minimize some measures in the resulting directed graph, for instance see [6,8]. On the other hand, there are many ways to color the vertices of graphs properly. A proper vertex coloring of a digraph D is defined, simply a vertex coloring of its underlying graph G. The chromatic number of a digraph provides interesting information about its subdigraphs. For instance, a theorem of Gallai proves that digraphs with high chromatic number always have long directed paths [9].

1.1. Background

Venkateswaran [16] initiated the study of the problem of orienting the edges of a given simple graph so that the maximum indegree of vertices is minimized. Afterwards, Asahiro et al. in [3] generalized this problem for weighted graphs. It was proved that, this problem can be solved in polynomial-time if all the edge weights are identical [3,14,16], but it is **NP**-hard in general [3]. Furthermore, the problem can be solved in polynomial-time if the input graph is a tree, but for planar bipartite graphs it is **NP**-hard [3]. For more information about the recent results about this problem see [2].

On the other hand, in 2004 Karoński, Łuczak and Thomason initiated the study of proper labeling [12]. They introduced an edge-labeling which is additive vertex-coloring that means for every edge uv, the sum of labels of the edges incident to u is different from the sum of labels of the edges incident to v [12]. Also, it is conjectured that three labels $\{1,2,3\}$ are sufficient for every connected graph, except K_2 (1, 2, 3-Conjecture, see [12]). This labeling have been studied extensively by several authors, for instance see [1,11]. Afterwards, Borowiecki et al. consider the directed version of this problem. Let D be a simple directed graph and suppose that each edge of D is assigned an integer label. For a vertex v of D, let $q^+(v)$ and $q^-(v)$ be the sum of labels lying on the arcs outgoing form v and incoming to v, respectively. Let $q(v)=q^+(v)-q^-(v)$.

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Borowiecki et al. proved that there is always a labeling from $\{1,2\}$, such that $q(v)$ is a proper coloring of D [4].

Furthermore, Borowiecki et al. consider another version of above problems and they show that every undirected graph G can be oriented so that adjacent vertices have different in-degrees [4]. In this work, we consider the problem of orienting the edges of an undirected graph such that adjacent vertices have different in-degrees and the maximum indegree of vertices is minimized in the resulting directed graph.

1.2. Notation

In this paper we consider simple graphs and we refer to [18] for standard notation and concepts. For a graph G, we use n and m to denote its numbers of vertices and edges, respectively. Also, for every $v \in V(G)$, $d(v)$ and $N_G(v)$ denote the degree of v and the neighbor set of v, respectively ($N_G(v) = \{u\colon uv \in E(G)\}$). We say that a set of vertices are *independent* if there is no edge between these vertices. The *independence number*, $\alpha(G)$ of a graph G is the size of a largest independent set of G. Also a *clique* of a graph is a set of mutually adjacent vertices. We denote the maximum degree of a graph G by $\Delta(G)$. A *directed graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges, with an incidence function D that associates with each edge of G an ordered pair of vertices of G. If $e = uv$ is an edge and $D(e) = u \to v$, then e is from u to v. The vertex u is the tail of e, and the vertex v its head. Note that every orientation D of a graph, introduced a digraph. The indegree $d_D^-(v)$ of a vertex v in D is the number of edges with head v of v.

For $k \in \mathbb{N}$, a *proper vertex k-coloring* of G is a function $c: V(G) \longrightarrow \{1,\dots,k\}$, such that if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. The smallest integer k such that G has a proper vertex k-coloring is called the *chromatic number* of G and denoted by $\chi(G)$. Similarly, for $k \in \mathbb{N}$, a *proper edge k-coloring* of G is a function $c: E(G) \longrightarrow \{1,\dots,k\}$, such that if $e, e' \in E(G)$ share a common endpoint, then $c(e)$ and $c(e')$ are different. The smallest integer k such that G has a proper edge k-coloring is called the *chromatic index* of G and denoted by $\chi'(G)$. By Vizing's theorem [17], the chromatic index of a graph G is equal to either $\Delta(G)$ or $\Delta(G)+1$. Those graphs G for which $\chi'(G) = \Delta(G)$ are said to belong to *Class* 1, and the other to *Class* 2. For a graph $G = (V, E)$, the *line graph* G is denoted by $L(G)$, is a graph with the set of vertices $E(G)$ and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. For simplicity and with a slight abuse of notation, we also denote by D the digraph resulting from an orientation D of the graph G.

1.3. Our results

The *proper orientations* of G are orientations D of G such that for every two adjacent vertices v and u, $d_D^-(v) \neq d_D^-(u)$. The *proper orientation number* of G is defined as $\overrightarrow{\chi}(G) = \min_{D\in\Gamma} \max_{v\in V(G)} d_D^-(v)$, where Γ is the set of proper orientations of G.

The proper orientation number is well-defined and every proper orientation of a graph G introduces a proper vertex coloring for its vertices. Thus, $\chi(G) - 1 \leqslant \overrightarrow{\chi}(G)$. On the other hand $\overrightarrow{\chi}(G) \leqslant \Delta(G)$. Consequently,

$$\chi(G) - 1 \leqslant \overrightarrow{\chi}(G) \leqslant \Delta(G). \tag{1}$$

In this work, we focus on regular graphs and planar graphs. We show that there is a polynomial-time algorithm for determining the proper orientation number of 3-regular graphs. But it is **NP**-complete to decide whether the proper orientation number of a given 4-regular graph is 3 or 4.

Theorem 1. *Determining the proper orientation number of a given* 4*-regular graph is* **NP***-hard; but there is a polynomial-time algorithm to determine the proper orientation number for* 3*-regular graphs.*

It is easy to see that $\overrightarrow{\chi}(G) = 1$ if and only if every connected component of G is a star. But for $\overrightarrow{\chi}(G) = 2$, we have the following:

Theorem 2. *It is* **NP***-complete to decide $\overrightarrow{\chi}(G) = 2$, for a given planar graph G.*

2. Regular graphs

Let G be an r-regular graph and suppose that D is a proper orientation of G with maximum indegree $\overrightarrow{\chi}(G)$. We have $\overrightarrow{\chi}(G) > \frac{1}{n}\sum_{v\in V(G)} d_D^-(v) = \frac{rn/2}{n}$. So we have the following simple observation about regular graphs.

Observation 1. For every r-regular graph G with $r \neq 0$, $\overrightarrow{\chi}(G) \geqslant \lceil \frac{r+1}{2} \rceil$.

Remark 1. By Observation 1, $\overrightarrow{\chi}(K_{2n,2n}) \geqslant n+1$. Let (X, Y) be the two parts of the complete bipartite graph $K_{2n,2n}$. By Kőnig's theorem [18], $K_{2n,2n}$ has a decomposition into $2n$ perfect matchings. Orient the edges of $n+1$ perfect matchings from X to Y, and other edges from Y to X. It is easy to see that this is a proper orientation with maximum indegree $n+1$. Therefore there is no absolute bound on the proper orientation number of bipartite graphs.

Here, we present a lemma about $(2k+1)$-regular graphs, then we use it to prove Theorem 1.

Lemma 1. *For every $(2k+1)$-regular graph G, $k \in \mathbb{N}$,*

(i) $\overrightarrow{\chi}(L(G)) = 3k$ *if and only if G belongs to Class* 1.
(ii) $\overrightarrow{\chi}(G) = k+1$ *if and only if* $\chi(G) = 2$.

Proof. (i) First, let G be a $(2k+1)$-regular graph belonging to *Class* 1. G has a proper edge coloring c, such that $c: E(G) \to \{1,\dots,2k+1\}$. Let $\{e_1,\dots,e_m\}$ be the set of edges of G and $V(L(G)) = E(G)$. Now, we present a proper orientation for $L(G)$ with maximum indegree $3k$. For every edge $e_ie_j \in E(L(G))$, such that $c(e_j) - c(e_i) > k$, orient e_ie_j from e_i to e_j. Also for every two numbers $p, q \in \{1,\dots,2k+1\}$,