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Finding outer-connected dominating sets in interval graphs ☆

Chiou-Jiun Lin [a], Jia-Jie Liu [b], Yue-Li Wang [a,*]

[a] Department of Information Management, National Taiwan University of Science and Technology, Taipei, Taiwan
[b] Department of Information Management, Shih Hsin University, Taipei, Taiwan

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ABSTRACT

An outer-connected dominating set in a graph $G = (V, E)$ is a set $S \subseteq V$ such that each vertex not in S is adjacent to at least one vertex in S and the subgraph induced by $V \setminus S$ is connected. In Keil and Pradhan (2013) [6], Keil and Pradhan gave a linear-time algorithm for finding a minimum outer-connected dominating set in a proper interval graph. In this paper, we generalize their result to find a minimum outer-connected dominating set in an interval graph in linear time.

1. Introduction

All graphs considered in this paper are simple, finite, and undirected. Let $G = (V, E)$ be a connected graph with vertex set V and edge set E, where $|V| = n$ and $|E| = m$. For any vertex $v \in V$, the *open neighborhood* of v is the set $N_G(v) = \{u \in V : uv \in E\}$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. For simplicity, $N_G(v)$ and $N_G[v]$ are simply written as $N(v)$ and $N[v]$, respectively, if no confusion is possible.

Definition 1.1. A set $D \subseteq V$ is a *dominating set* if every vertex not in D is adjacent to (is dominated by) at least one vertex in D. A dominating set S is an *outer-connected dominating set*, abbreviated as *OCD*-set, if the subgraph induced by $V \setminus S$ is connected. The *outer-connected domination number*, denoted by $\tilde{\gamma}_c(G)$, is the cardinality of a minimum *OCD*-set of G. The *outer-connected domination problem* is to find a minimum *OCD*-set of a graph G.

The outer-connected domination problem in graphs was introduced by Cyman in [3] and subsequently studied in [1,5,6,8]. The outer-connected domination problem is NP-complete for bipartite graphs [3], doubly chordal graphs [6], and undirected path graphs [6].

A graph $G = (V, E)$ is an *interval graph* if $V = \{1, 2, \ldots, n\}$ and there is an interval representation $\mathcal{I}(G)$ such that each vertex i in V corresponds to an interval I_i in $\mathcal{I}(G)$ and $(i, j) \in E$ if and only if I_i and I_j intersect in the interval representation. We also use $\mathcal{I}$ to denote $\mathcal{I}(G)$ when only one graph is mentioned. There exists an $O(n + m)$-time algorithm to recognize an interval graph and construct its interval model by using PQ-trees [2]. A *proper interval graph* is an interval graph that has an intersection model in which no interval properly contains another. In [6], Keil and Pradhan presented a linear-time algorithm for finding a minimum OCD-set in a proper interval graph. In this paper, we generalize their result to find a minimum outer-connected dominating set in an interval graph in linear time.

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* Corresponding author at: Department of Information Management, National Taiwan University of Science and Technology, Taipei, Taiwan.
E-mail address: ylwang@cs.ntust.edu.tw (Y.-L. Wang).

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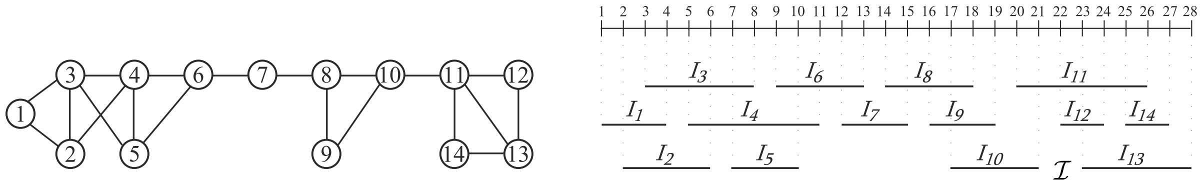

(a) an interval graph G (b) its interval representation $\mathcal{I}(G)$

Fig. 1. An interval graph G and its corresponding $\mathcal{I}(G)$.

The rest of this paper is organized as follows. In Section 2, some preliminaries are introduced. We also describe the reason why the algorithm proposed by Keil and Pradhan in [6] cannot be applied to finding a minimum OCD-set in interval graphs. In Section 3, we introduce our proposed algorithm for solving the outer-connected domination problem in interval graphs. Finally, concluding remarks are given in Section 4.

2. Preliminaries

Given an interval graph $G = (V, E)$, assume that $\mathcal{I}$ is its corresponding interval representation. Let I_i corresponding to $i \in V$ be an interval in $\mathcal{I}$. Let a_i and b_i be the two endpoints of I_i on a single line L, where $a_i < b_i$ for $1 \leqslant i \leqslant n$. Accordingly, let $I_i = [a_i, b_i]$. Without loss of generality, we may assume that no two distinct intervals in $\mathcal{I}$ have the same endpoints. Thus line L can be labeled with consecutive integers $1, 2, \ldots, 2n$ from left to right such that $\bigcup_{i=1}^{n}\{a_i, b_i\} = \{1, 2, \ldots, 2n\}$. Assume also that all intervals are labeled in increasing order of their left endpoints, namely $a_1 < a_2 < \cdots < a_n$. Figs. 1(a) and 1(b) depict an interval graph and its corresponding interval representation $\mathcal{I}$, respectively.

A vertex $v \in V$ is a *simplicial vertex* of G if $N[v]$ is a clique of G. An ordering $\alpha = (v_1, v_2, \cdots, v_n)$ is a *perfect elimination ordering* (PEO for short) of G if v_i is a simplicial vertex of $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for all i, $1 \leqslant i \leqslant n$, where $G[S]$ denotes the subgraph induced by the vertices in S. A PEO $\alpha = (v_1, v_2, \ldots, v_n)$ of a chordal graph is a *bicompatible elimination ordering* (BCO for short) if $\alpha^{-1} = (v_n, v_{n-1}, \ldots, v_1)$, i.e., the reverse of α, is also a PEO of G.

Theorem 2.1. *(See [4].) A graph G has a BCO if and only if it is a proper interval graph.*

In [6], the algorithm proposed by Keil and Pradhan is based on BCO to traverse the vertices in a proper interval graph G. By Theorem 2.1, their algorithm cannot be applied to finding a minimum OCD-set in an interval graph.

A vertex v is a *cut vertex* of a connected graph G if the number of connected components is increased after removing v.

Observation 2.2. *If vertex u is a non-cut vertex in an interval graph G, then the induced subgraph of $N(u)$ is connected.*

Definition 2.3. A cut vertex u is called a *chained cut-vertex* if the degree of vertex u is 2 and its two adjacent vertices are also cut vertices. A *chainless component* of G is a maximal subgraph of G in which there is no chained cut-vertex. The number of chainless components in G is denoted by $c(G)$. A vertex in a chainless component H is called a *chained end-vertex* if it is a chained cut-vertex in G.

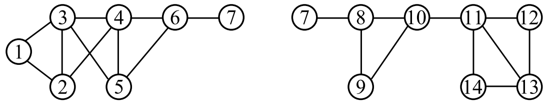

Fig. 2. Chainless components of the graph in Fig. 1.

Note that each chained end-vertex belongs to exactly two distinct chainless components. The set containing all chained end-vertices in chainless component H is the *chained-end set* of H, denoted by $\mathrm{CE}(H)$.

Example 1. We use the graph in Fig. 1 to illustrate chainless components. Since only vertex 7 is a chained cut-vertex, we have two chainless components. Thus $c(G) = 2$ (see Fig. 2).

Lemma 2.4. *For any OCD-set M of a graph G, all vertices in $V \setminus M$ are in the same chainless component.*

Proof. Suppose to the contrary that there exist two vertices $u, v \in V \setminus M$ which are in two different chainless components H_i and H_j, respectively. Clearly, every cut vertex in any path from u to v cannot be in M; otherwise, vertices u and v will be disconnected after the vertices in M are removed. Let s be a chained cut-vertex in a path from u to v, and r and t be its two adjacent cut vertices. Since s, r, and t are not in M, this implies that s is not dominated by any vertex in M, a contradiction. This completes the proof. □

Definition 2.5. A *relaxation outer-connected dominating set*, abbreviated as *ROCD*-set, of a chainless component H is a set $S \subseteq V(H)$ such that every vertex in $V(H) \setminus (S \cup \mathrm{CE}(H))$ is adjacent to at least one vertex in S and the subgraph induced by $V(H) \setminus S$ is connected. The *relaxation outer-connected domination number*, denoted by $\tilde{\gamma}_{rc}(H)$, is the cardinality of a minimum *ROCD*-set of H.

Note that, in an ROCD-set, all chained end-vertices might not be dominated; however, each of them can be an element in an ROCD-set.

Lemma 2.6. *For a graph G, it holds that* $\tilde{\gamma}_c(G) = \min_{1 \leqslant i \leqslant c(G)}\{|V(G) \setminus V(H_i)| + \tilde{\gamma}_{rc}(H_i)\}$.