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Bounds on locating total domination number of the Cartesian product of cycles and paths

Huaming Xing [a], Moo Young Sohn [b,*]

[a] Institute of Mathematics, Langfang Normal College, Langfang, 065000, PR China
[b] Mathematics Changwon National University, Changwon, 641-773, Republic of Korea

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ABSTRACT

The problem of placing monitoring devices in a system in such a way that every site in the safeguard system (including the monitors themselves) is adjacent to a monitor site can be modeled by total domination in graphs. Locating-total dominating sets are of interest when the intruder/fault at a vertex precludes its detection in that location. A total dominating set S of a graph G with no isolated vertex is a locating-total dominating set of G if for every pair of distinct vertices u and v in $V - S$ are totally dominated by distinct subsets of the total dominating set. The locating-total domination number of a graph G is the minimum cardinality of a locating-total dominating set of G. In this paper, we study the bounds on locating-total domination numbers of the Cartesian product $C_m \square P_n$ of cycles C_m and paths P_n. Exact values for the locating-total domination number of the Cartesian product $C_3 \square P_n$ are found, and it is shown that for the locating-total domination number of the Cartesian product $C_4 \square P_n$ this number is between $\lceil \frac{3n}{2} \rceil$ and $\lceil \frac{3n}{2} \rceil + 1$ with two sharp bounds.

1. Introduction

The location of monitoring devices, such as surveillance cameras or fire alarms, used to safeguard a system serves as the motivation for this work. The problem of placing monitoring devices in a system in such a way that every site in the system (including the monitors themselves) is adjacent to a monitor site can be modeled by total domination in graphs. Applications where it is also important that if there is a problem at a facility, the location can be uniquely identified by the set of monitors, can be modeled by a combination of total-domination and locating sets. Locating-total dominating set in graph was introduced by Haynes and Henning [5] and has been studied in [1,5–7] and elsewhere.

Graph theory terminology not presented here can be found in [3,4]. All graphs considered in this paper are simple without isolated vertices. Let $G = (V, E)$ be a graph with vertex set V and edge set E. For any vertex $v \in G$, the *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. We denote the degree of a vertex v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the text. For any $S \subseteq V$, $N(S) = \cup_{v \in S} N(v)$. Let $\langle S \rangle$ denote the graph induced by S. For two vertices $u, v \in V$, the *distance* between u and v is $d(u, v)$. The distance between a vertex u and a set S of vertices in a graph is defined as $d(u, S) = min\{d(u, v) | v \in S\}$. If S and T are two vertex disjoint subsets of V, then we denote the number of all edges of G that join a vertex of S and a vertex of T by $e[S, T]$. Throughout this paper, we use C_n and P_n to denote a cycle and a path of the order n, respectively.

For graphs G_1 and G_2, the *Cartesian product* $G_1 \square G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ where two ver-

* Corresponding author.
E-mail address: mysohn@changwon.ac.kr (M.Y. Sohn).

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tices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$.

Let $\{v_{ij} | 1 \le i \le m, 1 \le j \le n\}$ be the vertex set of $G = C_m \Box P_n$ so that the subgraph induced by $\mathcal{H}_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ is isomorphic to the path P_n for each $1 \le i \le m$ and the subgraph induced by $\mathcal{V}_j = \{v_{1j}, v_{2j}, \ldots, v_{mj}\}$ is isomorphic to the cycle C_m for each $1 \le j \le n$.

A subset $S \subseteq V$ is a *total dominating set* (abbreviated, TDS) if every vertex in V has a neighbor in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Total domination was introduced by Cockayne et al. [2] and is now well-studied in graph theory [3,4].

A total dominating set S in a graph $G = (V, E)$ is a *locating-total dominating set* (abbreviated, LTDS) of G if for every pair of distinct vertices u and v in $V - S$, $N(u) \cap S \neq N(v) \cap S$. The minimum cardinality of a locating-total dominating set is the *locating-total domination number* $\gamma_t^L(G)$. We call a $\gamma_t^L(G)$-set a locating-total dominating set in G of cardinality $\gamma_t^L(G)$.

In [5], Haynes et al. gave a lower bound on the locating-total domination number of a tree in terms of order and characterized the extremal tree as achieving equality in the lower bound. In [1], Chen and Sohn established lower and upper bounds on the locating-total domination number of trees and constructively characterized the extremal trees achieving the bounds. In [6], Henning and Löwenstein showed that the locating-total domination number of a claw-free cubic graph is at most one-half its order and characterized the graphs that achieved this bound. In [7], Henning and Rad gave lower and upper bounds on the locating-total domination number of a graph, showed that the locating-total domination number and total domination number of a connected cubic graph can differ significantly, and investigated the locating-total domination number of grid graph $P_m \Box P_n$ for small m.

In this paper, we establish upper bounds on locating total domination numbers of the Cartesian product $C_m \Box P_n$ of cycles C_m and paths P_n. In particular, we prove that for any positive integer n, $\gamma_t^L(C_3 \Box P_n) = n + 1$ and $\lceil \frac{3n}{2} \rceil \le \gamma_t^L(C_4 \Box P_n) \le \lceil \frac{3n}{2} \rceil + 1$, and these bounds are sharp.

2. Bounds of locating-total domination number of $C_m \Box P_n$

In this section, we present upper and lower bounds on the locating-total domination number of the Cartesian product of cycles C_m and paths P_n.

Theorem 2.1. *For any positive integers m, n such that $m \equiv 0 (mod\ 3)$ and $n \ge 3$, $\gamma_t^L(C_m \Box P_n) \le \frac{1}{3}m(n+1)$.*

Proof. Let $G \cong C_m \Box P_n$, where $m = 3t$ for a positive integer t. For any integers i, j such that $1 \le i \le 3$ and $1 \le j \le n$, let $D_{ij} = \mathcal{V}_j - \cup_{l=0}^{t-1}\{v_{(3l+i)j}\}$.

If $n = 3$, then $S = D_{11} \cup D_{23}$ is a LTDS of order $4t = \frac{1}{3}m(n+1)$ in G. If $n = 4$, then $S = D_{11} \cup D_{23} \cup (\cup_{i=0}^{t-1}\{v_{(3i+1)4}\})$ is a LTDS of order $5t = \frac{1}{3}m(n+1)$ in G. If $n = 5$, then $S = D_{11} \cup D_{23} \cup D_{35}$ is a LTDS of order $6t = \frac{1}{3}m(n+1)$ in G.

Assume that $n \ge 6$. Let $n = 6k + r$, where $k \ge 1$ and $0 \le r \le 5$. Let $S_0 = \cup_{j=0}^{k-1}(D_{1(6j+1)} \cup D_{2(6j+3)} \cup D_{3(6j+5)})$. Then $|S_0| = 6kt$.

If $r = 0$, then $S = S_0 \cup (\cup_{i=0}^{t-1}\{v_{(3i+2)n}\})$ is a LTDS of order $tn + t = \frac{1}{3}m(n+1)$ in G. If $r = 1$, then $S = S_0 \cup D_{1n}$ is a LTDS of order $6kt + 2t = \frac{1}{3}m(n+1)$ in G. If $r = 2$, then $S = S_0 \cup D_{1(n-1)} \cup (\cup_{i=0}^{t-1}\{v_{(3i+2)n}\})$ is a LTDS of order $6kt + 3t = \frac{1}{3}m(n+1)$ in G. If $r = 3$, then $S = S_0 \cup D_{1(n-2)} \cup D_{2n}$ is a LTDS of order $6kt + 4t = \frac{1}{3}m(n+1)$ in G. If $r = 4$, then $S = S_0 \cup D_{1(n-3)} \cup D_{2(n-1)} \cup (\cup_{i=0}^{t-1}\{v_{(3i+1)n}\})$ is a LTDS of order $6kt + 5t = \frac{1}{3}m(n+1)$ in G. If $r = 5$, then $S = S_0 \cup D_{1(n-4)} \cup D_{2(n-2)} \cup D_{3n}$ is a LTDS of order $6(k+1)t = \frac{1}{3}m(n+1)$ in G.

Therefore, $\gamma_t^L(G) \le |S| = \frac{1}{3}m(n+1)$. This completes the proof.

Theorem 2.2. *For any positive integers m, n such that $m \equiv 1 (mod\ 3)$ and $n \ge 3$,*

$$\gamma_t^L(C_m \Box P_n) \le \begin{cases} 11, & m = 4,\ n = 7; \\ 2m, & n = 5; \\ \frac{1}{3}(m-1)(n+1) + \lceil \frac{n}{2} \rceil, & \textit{otherwise}. \end{cases}$$

Proof. Let $G \cong C_m \Box P_n$, where $m = 3t + 1$ for a positive integer t. For any integers i, j such that $1 \le i \le 4$ and $1 \le j \le n$, let $D_{ij} = \mathcal{V}_j - \cup_{l=0}^{t-1}\{v_{(3l+i)j}\}$ and let $\lambda = \frac{1}{3}(m-1)(n+1) + \lceil \frac{n}{2} \rceil$.

If $n = 3$, then $S = D_{11} \cup D_{23}$ is a LTDS of order $4t + 2 = \lambda$ in G. If $n = 4$, then $S = D_{11} \cup D_{23} \cup (\cup_{i=0}^{t-1}\{v_{(3i+1)4}\})$ is a LTDS of order $5t + 2 = \lambda$ in G. If $n = 5$, then $S = \mathcal{V}_2 \cup \mathcal{V}_4$ is a LTDS of order $2m$ in G. If $m = 4$ and $n = 7$, then $S = D_{42} \cup \mathcal{V}_6 \cup \{v_{11}, v_{34}, v_{44}, v_{15}\}$ is a LTDS of order 11 in G.

Assume that $n \ge 6$. Let $n = 6k + r$, where $k \ge 1$ and $0 \le r \le 5$. Let $S_0 = \cup_{j=0}^{k-1}(D_{1(6j+1)} \cup D_{2(6j+3)} \cup D_{3(6j+5)})$. Then $|S_0| = 6kt + 3k$.

If $r = 0$, then $S = S_0 \cup (\cup_{i=0}^{t-1}\{v_{(3i+2)n}\})$ is a LTDS of order $6kt + 3k + t = \lambda$ in G. If $r = 1$, then $S = S_0 \cup D_{1n}$ is a LTDS of order $6kt + 3k + 2t + 1 = \lambda$ in G. If $r = 2$, then $S = S_0 \cup D_{1(n-1)} \cup (\cup_{i=0}^{t-1}\{v_{(3i+2)n}\})$ is a LTDS of order $6kt + 3k + 3t + 1 = \lambda$ in G. If $r = 3$, then $S = S_0 \cup D_{1(n-2)} \cup D_{2n}$ is a LTDS of order $6kt + 3k + 4t + 2 = \lambda$ in G. If $r = 4$, then $S = S_0 \cup D_{1(n-3)} \cup D_{2(n-1)} \cup (\cup_{i=0}^{t-1}\{v_{(3i+1)n}\})$ is a LTDS of order $6kt + 3k + 5t + 2 = \lambda$ in G. If $r = 5$, then $S = S_0 \cup D_{1(n-4)} \cup D_{2(n-2)} \cup D_{3n}$ is a LTDS of order $6kt + 3k + 6t + 3 = \lambda$ in G. This completes the proof.

Theorem 2.3. *For any positive integers m, n such that $m \equiv 2 (mod\ 3)$ and $n \ge 3$,*

$$\gamma_t^L(C_m \Box P_n) \le \begin{cases} \frac{2}{5}mn, & n = 5 \text{ or } m = 5, \\ & n \equiv 0 (mod\ 5); \\ \frac{1}{3}(m+1)(n+1) - 1, & \textit{otherwise}. \end{cases}$$

Proof. Let $G \cong C_m \Box P_n$, where $m = 3t + 2$ for a positive integer t. For any integers i, j such that $1 \le i \le 5$ and $1 \le j \le n$, let $D_{ij} = \mathcal{V}_j - \cup_{l=0}^{t-1}\{v_{(3l+i)j}\}$ and let $\mu = \frac{1}{3}(m+1)(n+1) - 1$.

If $n = 3$, then $S = (D_{11} - \{v_{m1}\}) \cup D_{33}$ is a LTDS of order $4t + 3 = \mu$ in G. If $n = 4$, then $S = D_{11} \cup D_{23} \cup$