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Rainbow 2-Connection Numbers of Cayley Graphs ☆

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ABSTRACT

A path in an edge colored graph is said to be a rainbow path if no two edges on this path share the same color. For an l-connected graph Γ and an integer k with $1 \leq k \leq l$, the rainbow k-connection number of Γ is the minimum number of colors required to color the edges of Γ such that any two distinct vertices of Γ are connected by k internally disjoint rainbow paths. In this paper, a method is provided for bounding the rainbow 2-connection numbers of graphs with certain structural properties. Using this method, we consider the rainbow 2-connection numbers of Cayley graphs, especially, those defined on abelian groups and dihedral groups.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of [2] for those not described here.

For a graph Γ, we denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set of Γ, respectively. An *edge-coloring* of a graph Γ is a mapping from $E(\Gamma)$ to some finite set of colors. A path in an edge colored graph is said to be a *rainbow path* if no two edges on this path share the same color. Let Γ be an edge colored l-connected graph, where l is a positive integer. For $1 \leq k \leq l$, the graph Γ is *rainbow k-connected* if any two distinct vertices of Γ are connected by k internally disjoint rainbow paths, while the coloring is called a *rainbow k-coloring*. The *rainbow k-connection number* of Γ, denoted by $rc_k(\Gamma)$, is the minimum number of colors required to color the edges of Γ to make the graph rainbow k-connected. For simplicity, we write $rc(\Gamma)$ for $rc_1(\Gamma)$ and call it *rainbow connection number*. A well-known theorem of Menger [14] shows that in every l-connected graph Γ with $l \geq 1$, there exist k internally disjoint paths connecting every two distinct vertices u and v for every integer k with $1 \leq k \leq l$. By coloring the edges of Γ with distinct colors, we know that every two distinct vertices of Γ are connected by k internally disjoint rainbow paths, and thus the function $rc_k(\Gamma)$ is well-defined for every $1 \leq k \leq l$. An easy observation is that $rc_k(\Gamma) \leq rc_k(\Sigma)$ for each l-connected spanning subgraph Σ of the graph Γ. We note also the trivial fact that if C_n is a cycle with $n \geq 3$, then $rc_2(\mathsf{C}_n) = n$.

The concept of rainbow k-connection number was first introduced by Chartrand et al. ([3] for $k = 1$, and [4] for general k). Since then, a considerable amount of research has been carried out towards the function $rc_k(\Gamma)$, see [12] for a survey on this topic. Chartrand et al. [4] proved that for every integer $k \geq 2$, there exists an integer $f(k)$ such that if $n \geq f(k)$, then $rc_k(\mathsf{K}_n) = 2$. With a similar method, Li and Sun [11] obtained that for every integer $k \geq 2$, there exists an integer $g(k) = 2k\lceil \frac{k}{2} \rceil$ such that $rc_k(\mathsf{K}_{n,n}) = 3$ for any $n \geq g(k)$. Fujita et al. [6] and He et al. [8] investigated the rainbow k-connection number of random graphs. In particular, it was shown in [10] that if Γ is a 2-connected graph with n vertices, then $rc_2(\Gamma) \leq n$ with equality if and only if Γ is a cycle of order n.

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Let G be a finite group with identity element 1. Let S be a subset of G such that $1 \notin S = S^{-1} := \{s^{-1} \mid s \in S\}$. The *Cayley graph* $\mathrm{Cay}(G,S)$ is defined on G such that two 'vertices' g and h are adjacent if and only if $g^{-1}h \in S$. Hence $\mathrm{Cay}(G,S)$ is a well-defined simple regular graph of valency $|S|$. It is well-known that $\mathrm{Cay}(G,S)$ is connected if and only if S is a generating set of G. In a Cayley graph $\mathrm{Cay}(G,S)$, an edge $\{g,h\}$ is called an *s-edge* if $g^{-1}h$ or $h^{-1}g$ equals some s in S.

Cayley graphs have been an active topic in algebraic graph theory for a long time. Actually, interconnection networks are often modeled by highly symmetric Cayley graphs [1]. The rainbow connection number of a graph can be applied to measure the safety of a network. Thus the object of the rainbow connection numbers of Cayley graphs should be meaningful. Li et al. [9], Lu and Ma [13] discussed the rainbow connection numbers of Cayley graphs. This motivates us to consider the rainbow 2-connection numbers of Cayley graphs. In this paper, we establish a lemma for bounding the rainbow 2-connection numbers of graphs satisfying certain structural properties. Using this lemma, we consider the rainbow 2-connection numbers of Cayley graphs, especially, those defined on abelian groups and on dihedral groups.

2. Rainbow 2-connection numbers of Cayley graphs

Let Γ be a graph. For $U, V \subseteq V(\Gamma)$, we denote by $\Gamma[U,V]$ the subgraph on $U \cup V$ with edge set $\{\{u,v\} \in E(\Gamma) \mid u \in U, v \in V\}$. For a partition $\mathcal{B} = \{U_0, U_1, \cdots, U_{m-1}\}$ of $V(\Gamma)$, define a graph $\Gamma_{\mathcal{B}}$ with vertex set $\mathcal{B}$ such that $U_i, U_j \in \mathcal{B}$ are adjacent in $\Gamma_{\mathcal{B}}$ if and only if some $u \in U_i$ is adjacent to some $v \in U_j$ in Γ. The graph $\Gamma_{\mathcal{B}}$ is called a *quotient graph* of Γ. The following technical lemma is very important.

Lemma 2.1. *Let Γ be a 2-connected graph. Assume that $V(\Gamma)$ has a partition $\mathcal{B} = \{U_0, U_1, \cdots, U_{m-1}\}$ such that $\Gamma_{\mathcal{B}}$ is 2-connected, and for each i, the subgraph $\Gamma[U_i, U_i]$ is 2-connected.*

(i) *Suppose that for each pair of adjacent vertices U_i and U_j in $\Gamma_{\mathcal{B}}$, the subgraph $\Gamma[U_i, U_j]$ has no isolate vertices. Then*

$$rc_2(\Gamma) \le \max\{rc_2(\Gamma[U_i, U_i]) \mid 0 \le i < m\} + rc_2(\Gamma_{\mathcal{B}}).$$

(ii) *Suppose that $E(\Gamma[U_i, U_{i+1}]) \ne \emptyset$ for $0 \le i < m$, and every $u \in U_i$ is adjacent to some $v \in U_{i-1}$ or some $w \in U_{i+1}$ in Γ, reading the subscripts modulo m. Then*

$$rc_2(\Gamma) \le (\max\{rc_2(\Gamma[U_i, U_i]) \mid 0 \le i < m\} + 1)m.$$

Proof. Denote $\Gamma_i = \Gamma[U_i, U_i]$ and $c = \max\{rc_2(\Gamma_i) \mid 0 \le i < m\}$.

(i) Let C be a set of c colors and D be a set of $rc_2(\Gamma_{\mathcal{B}})$ colors with $C \cap D = \emptyset$. For $\Gamma_{\mathcal{B}}$, we choose a rainbow 2-coloring $\bar{\theta} : E(\Gamma_{\mathcal{B}}) \to D$. For each graph Γ_i, assign a rainbow 2-coloring $\theta_i : E(\Gamma_i) \to C$. Define an edge-coloring θ of Γ by

$$\theta(e) = \begin{cases} \theta_i(e) & \text{if } e \in E(\Gamma_i) \text{ for } 0 \le i < m; \\ \bar{\theta}(\{U_i, U_j\}) & \text{if } \{U_i, U_j\} \in E(\Gamma_{\mathcal{B}}) \text{ and} \\ & e \in E(\Gamma[U_i, U_j]). \end{cases}$$

Let u and v be any two distinct vertices of Γ. If u and v are contained in some Γ_i, then there exist two internally disjoint rainbow paths by means of the rainbow 2-coloring θ_i. Suppose $u \in V(\Gamma_i)$ and $v \in V(\Gamma_j)$ satisfying $i \ne j$. In the quotient graph $\Gamma_{\mathcal{B}}$, there exist two internally disjoint rainbow paths connecting U_i and U_j. Denote them by $U_i, U_{i_1}, U_{i_2}, \cdots, U_{i_s}, U_j$ and $U_i, U_{j_1}, U_{j_2}, \cdots, U_{j_t}, U_j$. Since U_{i_s} and U_j are adjacent in $\Gamma_{\mathcal{B}}$, by the assumptions, we know that the subgraph $\Gamma[U_{i_s}, U_j]$ has no isolate vertices. Then there exists a vertex $v_{i_s} \in U_{i_s}$ satisfying $v_{i_s}v \in E(\Gamma)$. Similarly, there exist some vertices $v_{i_r} \in U_{i_r}$ for $1 \le r \le s-1$ and $v_i \in U_i$ such that $v_{i_s}v_{i_{s-1}}, v_{i_{s-1}}v_{i_{s-2}}, \cdots, v_{i_2}v_{i_1}, v_{i_1}v_i \in E(\Gamma)$. Obviously, $\mathsf{P}' = u, \mathsf{P}^1, v_i, v_{i_1}, v_{i_2}, \cdots, v_{i_s}, v$ is a rainbow path connecting u and v, where P^1 is a rainbow path between u and v_i in Γ_i. Since U_i and U_{j_1} are adjacent in $\Gamma_{\mathcal{B}}$, by the assumptions, we have that the subgraph $\Gamma[U_i, U_{j_1}]$ has no isolate vertices. Thus there exists a vertex $v_{j_1} \in U_{j_1}$ satisfying $uv_{j_1} \in E(\Gamma)$. Similarly, there exist some vertices $v_{j_r} \in U_{j_r}$ for $2 \le r \le t$ and $v_j \in U_j$ such that $v_{j_1}v_{j_2}, v_{j_2}v_{j_3}, \cdots, v_{j_{t-1}}v_{j_t}, v_{j_t}v_j \in E(\Gamma)$. Obviously, $\mathsf{P}'' = u, v_{j_1}, v_{j_2}, \cdots, v_{j_t}, v_j, \mathsf{P}^2, v$ is also a rainbow path connecting u and v, where P^2 is a rainbow path between v_j and v in Γ_j. Note that P' and P'' are internally disjoint. Thus Γ is rainbow 2-connected with the edge-coloring θ, and so $rc_2(\Gamma) \le \max\{rc_2(\Gamma[U_i, U_i]) \mid 0 \le i < m\} + rc_2(\Gamma_{\mathcal{B}})$.

(ii) Consider the spanning subgraph Σ of Γ with edge set

$$E(\Sigma) = \left(\bigcup_{i=0}^{m-1} E(\Gamma_i)\right) \cup \left(\bigcup_{i=0}^{m-1} E(\Gamma[U_i, U_{i+1}])\right).$$

Since $E(\Gamma[U_i, U_{i+1}]) \ne \emptyset$ for $0 \le i < m$, we obtain that $\Sigma_{\mathcal{B}}$ is a cycle of length m. Let $C_0, C_1, \cdots, C_{m-1}$ be c-sets of colors such that $C_i \cap C_j = \emptyset$ if $0 \le i < j < m$. For each graph Γ_i, since $rc_2(\Gamma_i) \le c$, we assign a rainbow 2-coloring $\eta_i : E(\Gamma_i) \to C_i$. Choose m colors $c_1, c_2, \cdots, c_m$ which are not used above. Define an edge-coloring η of Σ as follows:

$$\eta(e) = \begin{cases} \eta_i(e) & \text{if } e \in E(\Gamma_i) \text{ for } 0 \le i < m; \\ c_i & \text{if } e \in E(\Gamma[U_{i-1}, U_i]) \text{ for } 1 \le i \le m. \end{cases}$$

Let u and v be any two distinct vertices of Γ. If u and v are contained in some U_i for $0 \le i \le m-1$, then there exist two internally disjoint rainbow paths connecting u and v by means of the rainbow 2-coloring η_i. Without loss of generality, we assume that $u \in U_i$ and $v \in U_j$ with $0 \le i \ne j \le m-1$. Then there also exist two internally disjoint rainbow paths connecting u and v since $\Sigma_{\mathcal{B}}$ is a cycle and the colors $c_1, c_2, \cdots, c_m$ are not used in Γ_i for $0 \le i \le m-1$. Hence Γ is rainbow 2-connected, and so part (ii) follows from enumerating the number of colors used for η. □

Let G be a group and N a normal subgroup of G. Then all (left) cosets of N in G form a group under the product