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Hamilton cycles in implicit claw-heavy graphs

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ABSTRACT

We define G to be implicit claw-heavy if every induced claw of G has a pair of nonadjacent vertices such that their implicit degree sum is at least $|V(G)|$. In this paper, we show that an implicit claw-heavy graph G is hamiltonian if we impose certain additional conditions on G involving forbidden induced subgraphs. Our result extends a previous theorem of Chen et al. (2009) [6] on the existence of hamiltonian cycles in claw-heavy graphs.

1. Introduction

In this paper, we consider only undirected, finite and simple graphs. For notation and terminology not defined here can be found in [3].

Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. For a vertex $u \in V(G)$, define $N_H(u) = \{v \in V(H) : uv \in E(G)\}$. The degree of u in H is denoted by $d_H(u) = |N_H(u)|$. If $H = G$, we can use $N(u)$ and $d(u)$ in place of $N_G(u)$ and $d_G(u)$, respectively.

A cycle in G is called a hamiltonian cycle if it contains all vertices of $V(G)$. And G is called hamiltonian if it contains a hamiltonian cycle. The hamiltonian cycle problem (determining whether a hamiltonian cycle exists in a given graph) is a classical problem in graph theory. Since the hamiltonian cycle problem is NP-complete, it is natural and very interesting to study sufficient conditions for the existence of hamiltonian cycles in graphs.

Dirac [7] pioneered the hamiltonian cycle problem study. He [7] proved that every graph is hamiltonian if each of its vertices has degree at least $|V(G)|/2$. This original result started a new approach to develop sufficient conditions on degrees for a graph to be hamiltonian. A lot of effort has been made by various scholars in generalization of Dirac's theorem. Ore [13] obtained that every graph is hamiltonian if each pair of its nonadjacent vertices has degree sum at least $|V(G)|$. Fan [8] proposed that only the pairs of vertices that are distance two are essential in Dirac's theorem. There have been several survey papers in hamiltonian cycle problems, see for example [10,11].

If H is a graph, then we say that a graph G is H-free if G contains no copy of H as an induced subgraph; the graph H will be also referred to in this paper as forbidden subgraphs. Specifically, the four-vertex star $K_{1,3}$ will be also called the claw and in this case we say that G is claw-free. Further graphs that will be often considered as forbidden subgraphs are shown in Fig. 1. Forbidden subgraph condition is another important type of conditions for the existence of hamiltonian cycles in graphs. Broersma and Veldman [5] proved that, every 2-connected claw-free graph G is hamiltonian if G is P_7-free and D-free. Faudree et al. [9] obtained a similar result as Broersma and Veldman's theorem. Precisely, they proved that, every 2-connected graph G is hamiltonian if G is claw-free, P_7-free and H-free.

Later, Broersma et al. [4] established another kind of conditions by restricting the numerical conditions to certain substructures; they proved that: Let G be a 2-connected graph. If every induced claw of G has at least two

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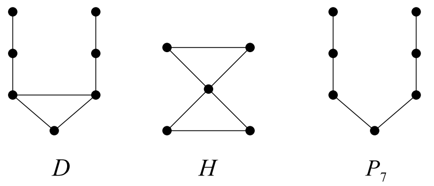

Fig. 1. Forbidden subgraphs.

nonadjacent vertices with degree at least $|V(G)|/2$, and moreover P_7-free and D-free, or P_7-free and H-free, then G is hamiltonian. In 2009, Chen et al. [6] relaxed "every induced claw of G has at least two nonadjacent vertices with degree at least $|V(G)|/2$" in [4] to "every induced claw of G has a pair of nonadjacent vertices with degree sum at least $|V(G)|$", and obtained a sufficient condition for a 2-connected graph to be hamiltonian. It is clear that every P_6 (a path on 6 vertices)-free graph is also P_7-free and D-free. So we can get a corollary from Chen's theorem that every 2-connected, claw-heavy and P_6-free graph is hamiltonian.

Dirac's condition [7] requires that every vertex has large degree. But in case that some vertices may have small degrees, we hope to use some large degree vertices to replace small degree vertices in the right position, so that we may construct a longer cycle. This idea leads to the definition of implicit degree given by Zhu, Li and Deng [14] in 1989. We use $N_2(v)$ to denote the vertices which are at distance 2 from v in G.

Definition 1. (See [14].) Let v be a vertex of a graph G and $d(v) = l + 1$. Set $M_2 = \max\{d(u) : u \in N_2(v)\}$. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, then let $d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_l \leq d_{l+1} \leq \ldots$ be the degree sequence of vertices of $N(v) \cup N_2(v)$. Define

$$d^*(v) = \begin{cases} d_{l+1}, & \text{if } d_{l+1} > M_2; \\ d_l, & \text{otherwise.} \end{cases}$$

Then the implicit degree of v is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then $id(v) = d(v)$.

Clearly, $id(v) \geq d(v)$ for every vertex v from the definition of implicit degree. In this paper, we consider the hamiltonian cycle problem on a 2-connected graph under implicit degree condition. Our main result is as follows.

Main result. Let G be a 2-connected graph. If every induced claw of G has a pair of nonadjacent vertices with implicit degree sum at least $|V(G)|$ and G is P_6-free, then G is hamiltonian.

Remark. The graph in Fig. 2 shows, our result does strengthen the result in [6]. Let $n \geq 12$ be an even integer and $K_{n/2-3} \cup K_{n/2}$ denote the union of two complete graphs $K_{n/2-3}$ and $K_{n/2}$. And let $V(K_{n/2-3}) = \{x_1, x_2, \ldots, x_{n/2-3}\}$ and $V(K_{n/2}) = \{y_1, y_2, \ldots, y_{n/2}\}$. We choose a graph G with $V(G) = V(K_{n/2-3} \cup K_{n/2}) \cup \{u, v, w\}$ and $E(G) = E(K_{n/2-3} \cup K_{n/2}) \cup \{x_i y_i, x_i y_{i+1} : i = 1, 2, \ldots, n/2-3\} \cup \{uw, vw, wy_{n/2-3}\} \cup \{ux_i, vx_i : i = 1, 2, \ldots, n/2-3\}$. It is easy to see that G is a hamiltonian graph not

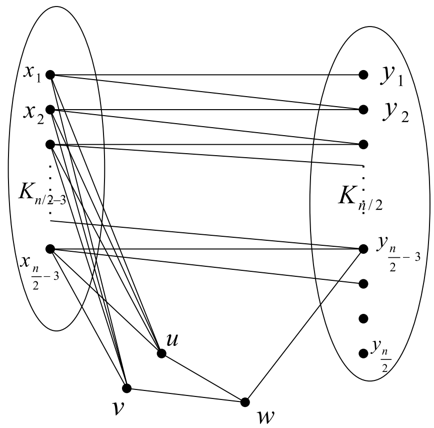

Fig. 2. Graph in Remark.

satisfying the conditions in [6]. But Since $id(u) = id(v) = n/2 - 1$ and $id(y_j) = n/2 + 1$ for each $j = 1, 2, \ldots, n/2$, G satisfies the conditions of our result.

2. Some properties of nonextendable cycles

For an induced claw of G with vertex set $\{u, v, w, x\}$ and edge set $\{uv, uw, ux\}$, we call u the center of the claw, and the vertices v, w, x end-vertices of the claw. Let G be a graph on n vertices, a vertex v of G is called implicit heavy if $id(v) \geq n/2$. If v is not implicit heavy, we call it implicit light. A claw of G is called implicit 2-heavy if at least two of its end-vertices are implicit heavy. And G is called implicit 2-heavy if all its induced claws are implicit 2-heavy. Moreover, G is called implicit claw-heavy if every induced claw of G has a pair of end-vertices x and y such that $id(x) + id(y) \geq n$. Clearly, every 2-heavy (claw-heavy) graph is implicit 2-heavy (implicit claw-heavy), every implicit 2-heavy graph is implicit claw-heavy, but an implicit claw-heavy graph is not necessarily implicit 2-heavy.

For a cycle C in G with a given orientation and a vertex x in C, x^+ and x^- denote the successor and the predecessor of x in C, respectively. And for any $I \subseteq V(C)$, let $I^- = \{x : x^+ \in I\}$ and $I^+ = \{x : x^- \in I\}$. For two vertices $x, y \in C$, xCy denotes the subpath of C from x to y. We use $y\bar{C}x$ for the path from y to x in the reversed direction of C. A similar notation is used for paths.

A cycle C is called implicit heavy if it contains all implicit heavy vertices of G; it is called extendable if there exists a longer cycle in G containing all vertices of C.

Lemma 1. *(See [1].) Let G be a 2-connected graph of order n and C be a nonextendable cycle of G with length at most $n-1$. If xPy is a path of G such that $V(C) \subset V(P)$ and $|V(P)| > |V(C)|$, then $d(x) + d(y) < n$.*

Lemma 2. *Let G be a 2-connected graph of order n and C be a nonextendable cycle of G with length at most $n-1$. If xPy is a path of G such that $V(C) \subset V(P)$ and $|V(P)| > |V(C)|$, then $xy \notin E(G)$ and $id(x) + id(y) < n$.*

Proof. Suppose to the contrary that $id(x) + id(y) \geq n$. Clearly, x and y have no common neighbor in $V(G) - V(P)$