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VC-dimension of perimeter visibility domains

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ABSTRACT

We obtain an upper bound of 7 for the VC-dimension of Perimeter Visibility Domains in simple polygons.

1. Introduction

The VC-dimension is a fundamental parameter of a range space that, intuitively speaking, measures how differently the ranges intersect with subsets of the ground set. Besides its importance in machine learning, VC-dimension became significant to computational geometry, chiefly through its role in the Epsilon Net Theorem by Haussler and Welzl [4]. This theorem states that whenever a range space has got finite VC-dimension d then there exists an ε-net for this space of size $Cd\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ for some small constant C.

For many important geometric range spaces such as rectangles, circles or halfspaces, the VC-dimension is not hard to estimate. The VC-dimension of range spaces of visibility domains was first considered by Kalai and Matoušek [6]. As an application, the Epsilon Net Theorem gives an upper bound of $O(r\log r)$ on the number of guards needed to guard a polygonal art gallery where every point sees at least an r-th part of the entire polygon. The VC-dimension of the set of visibility polygons inside polygons contributes a constant factor to this upper bound. Therefore, better upper bounds on the VC-dimension immediately yield better upper bounds on the number of guards needed. Kalai and Matoušek [6] showed that the VC-dimension of visibility polygons in a simple polygon is finite. They also gave an example of a gallery with VC-dimension 5. Furthermore, they showed that there is no constant that bounds the VC-dimension for polygons with holes. For simple polygons, Valtr [11] gave an example of a gallery with VC-dimension 6 and proved an upper bound of 23. In the same paper he showed an upper bound for the VC-dimension of a gallery with holes of $O(\log h)$ where h is the number of holes, and art galleries with holes that have VC-dimensions of this size can also be constructed. These results for galleries with holes easily carry over to the case of Perimeter Visibility Domains. In [3] Gilbers and Klein show that the VC-dimension of Visibility Polygons of a Simple Polygon is at most 14 (an extended abstract of this paper appeared in [2]). Isler et al. [5] examined the case of exterior visibility. In this setting the points of S lie on the boundary of a polygon P and the ranges are sets of the form $\mathrm{vis}(v)$ where v is a point outside the convex hull of P. They showed that the VC-dimension is 5. They also considered a more restricted version of exterior visibility where the view points v all must lie on a circle around P, with VC-dimension 2. For a 3-dimensional version of exterior visibility with S on the boundary of a polyhedron Q they found that the VC-dimension is in $O(\log n)$ where n is the number of vertices of Q. King [7] examined

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the VC-dimension of visibility regions on polygonal terrains. For 1.5-dimensional terrains he proved that the VC-dimension equals 4 and on 2.5-dimensional terrains there is no constant bound. Kirkpatrick [9] showed that it is possible to guard a polygon with $O(r \log \log r)$ many perimeter guards (i.e. guards on the boundary of the polygon) if every point on the boundary sees at least an r-th part of the boundary. In [8] King and Kirkpatrick extended this work and obtained an $O(\log \log \mathrm{OPT})$-approximation algorithm for finding the minimum number of guards on the perimeter that guard the polygon. As an open question they asked whether it is easier to find the VC-dimension in the case of perimeter guards than for general visibility polygons. They show that the corresponding VC-dimension is at least 5. In this paper we show that in the case of simple polygons one can obtain an upper bound of 7 for this VC-dimension, by extending the technique from Gilbers and Klein [1].

2. VC-dimension

The following definition of VC-dimension is adopted from [10].

Definition 1. Let $\mathcal{F}$ be a set system on a set X. A subset $S \subseteq X$ is said to be shattered by $\mathcal{F}$ if each of the subsets of S can be obtained as the intersection of some $F \in \mathcal{F}$ with S. We define the VC-dimension of $\mathcal{F}$, denoted by $\dim(\mathcal{F})$, as the supremum of the sizes of all finite shattered subsets of X. If arbitrarily large subsets can be shattered, the VC-dimension is ∞.

If a finite subset $Y \subseteq X$ with $|Y| = n$ is shattered by $\mathcal{F}$, then the set $\Pi_{\mathcal{F}}(Y) = \{Y \cap F \mid F \in \mathcal{F}\}$ has 2^n elements. For every such Y we define the *coarseness* of Y to be $c_{\mathcal{F}}(Y) = 2^n - |\Pi_{\mathcal{F}}(Y)|$. Obviously, Y is shattered by $\mathcal{F}$ iff $c_{\mathcal{F}}(Y) = 0$.

Let now Y be shattered by $\mathcal{F}$ and $\mathcal{F}$ be the union of $\mathcal{F}'$ and $\mathcal{F}''$. If $c_{\mathcal{F}'}(Y) = k > 0$ there are k subsets of Y that are not shattered by $\mathcal{F}'$. It is clear that for each of these subsets Z there must be some set $F'' \in \mathcal{F}''$ such that $Z = F'' \cap Y$. Therefore $\Pi_{\mathcal{F}''}(Y) \geq k$. We have just proven the following lemma.

Lemma 1. *Let Y be shattered by $\mathcal{F}$ and $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$. Then $\Pi_{\mathcal{F}''}(Y) \geq c_{\mathcal{F}'}(Y)$.*

We next reformulate a property of shattered sets that Gilbers and Klein already used in [1] and that will again be a cornerstone of our proof:

Lemma 2. *Let Y', Y'' be disjoint subsets of the finite set Y and $\mathcal{F}' \cup \mathcal{F}'' = \mathcal{F}$. Then $c_{\mathcal{F}}(Y) \geq c_{\mathcal{F}'}(Y') \cdot c_{\mathcal{F}''}(Y'')$.*

Proof. Let $Z' \subseteq Y'$, $Z'' \subseteq Y''$ such that for no $F' \in \mathcal{F}'$: $Z' = Y' \cap F'$ and for no $F'' \in \mathcal{F}''$: $Z'' = Y'' \cap F''$. Then there can be no $F \in \mathcal{F}$ such that $Z' \cup Z'' = Y \cap F$. That means that for every such combination of subsets of Y', Y'' there is a distinct subset of Y that does not have a representation as an intersection of Y with some $F \in \mathcal{F}$. There are $c_{\mathcal{F}'}(Y') \cdot c_{\mathcal{F}''}(Y'')$ such combinations. The inequality follows. □

Corollary 1. *Let Y', Y'' be disjoint subsets of the finite set Y and $\mathcal{F}' \cup \mathcal{F}'' = \mathcal{F}$. If $\mathcal{F}$ shatters Y then $\mathcal{F}'$ shatters Y', or $\mathcal{F}''$ shatters Y''.*

Another property that we will make use of is the following.

Lemma 3. *Let X be a set that is shattered by $\mathcal{F}$, $Y \subsetneq X$ and $\mathcal{F}_Y = \{F \in \mathcal{F} \mid Y \subseteq F\}$. Then $X \setminus Y$ is shattered by $\mathcal{F}_Y$.*

Proof. Suppose not. Then there is some $X' \subseteq X \setminus Y$ such that $F \cap X \setminus Y = X'$ for no $F \in \mathcal{F}_Y$. But then there can be no $F \in \mathcal{F}_Y$ with $X \cap F = X' \cup Y$. As there can also be no such set in $\mathcal{F} \setminus \mathcal{F}_Y$, X can not be shattered by $\mathcal{F}$. □

3. Perimeter Visibility Domains

Let P be a simple polygon with boundary B. As usual, for a point $p \in P$ its *Visibility Polygon* $\mathrm{vis}(p)$ is the set of points v such that the whole segment $\overline{pv}$ is contained in P. We restrict our attention to the portions of visibility polygons on the boundary, $\mathrm{vis}(p) \cap B$. For every $p \in B$ we will call this boundary portion its *Perimeter Visibility Domain* and denote it by $V(p)$. As we are only concerned with this kind of visibility domains in this paper, we will refer to them simply as *Visibility Domains*, below. We are now interested in the VC-dimension of the set system $\mathcal{V} = \{V(b)\}_{b \in B}$ on the set B.

Definition 2. A subset I of B is called an interval in B if there is a continuous injective function $\pi : [0, 1] \longrightarrow B$ with image I or if $I = \emptyset$.

Definition 3. Let B_1 be a subset of B. We call a subset B_2 of B *interval-like relative to* B_1, if its intersection with B_1 equals the intersection of B_1 with some interval in B.

For an illustration of this concept, see Fig. 1. The key geometric insight that we will need is formulated in the following lemma:

Lemma 4. *Let $a_1, a_2 \in B$ be two distinct boundary points such that $B \setminus \{a_1, a_2\}$ splits into two connected components C and D. Then for every point $c \in C$, $V(c)$ is interval-like relative to $D \cap V(a_1) \cap V(a_2)$.*

Proof. We have to find an interval I in B such that $I \cap D \cap V(a_1) \cap V(a_2)$ equals $V(c) \cap D \cap V(a_1) \cap V(a_2)$.

Let to this end $\pi : [0, 1] \longrightarrow D \cup \{a_1, a_2\}$ be a continuous bijective map with $\pi(0) = a_1$ and $\pi(1) = a_2$.

If the intersection $D \cap V(c)$ of D with the visibility domain of c is empty, we set $I = \emptyset$.

Otherwise there are values $f = \inf\{x \in (0, 1) : \pi(x) \in V(c)\}$ and $\ell = \sup\{x \in (0, 1) : \pi(x) \in V(c)\}$. We set I as the image of $[f, \ell]$ under π, $I = \pi[[f, \ell]]$, see Fig. 2.

It remains to show that the two intersections $I \cap D \cap V(a_1) \cap V(a_2)$ and $V(c) \cap D \cap V(a_1) \cap V(a_2)$ are indeed equal.

By the definition of I it is clear that $V(c) \cap D \cap V(a_1) \cap V(a_2) \subseteq I \cap D \cap V(a_1) \cap V(a_2)$, as $V(c) \cap D \subseteq I \cap D$.