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Fractional programming formulation for the vertex coloring problem

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ABSTRACT

We devise a new formulation for the vertex coloring problem. Different from other formulations, decision variables are associated with pairs of vertices. Consequently, colors will be distinguishable. Although the objective function is fractional, it can be replaced by a piece-wise linear convex function. Numerical experiments show that our formulation has significantly good performance for dense graphs.

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1. Introduction

The vertex coloring problem (VCP) is a well-known NP-hard [1] combinatorial optimization problem with a large number of applications including scheduling, register allocation, and timetabling (see the survey [2] for the details). In this problem, we are given a simple and undirected graph $G = (V, E)$. The objective is to find an assignment of colors to V such that no two adjacent vertices share the same color and the number of colors used is minimized.

In the standard formulation for VCP, letting C be a set of colors, we introduce a decision variable x_{vc} ($\forall v \in V$, $\forall c \in C$) which takes 1 if v receives color c and takes 0 otherwise. Since every graph can be colored with $n = |V|$ colors, it suffices to set $C = \{1, 2, \ldots, n\}$. Although this formulation is intuitive and simple, there exists a strong symmetry in the feasible region resulting from the indistinguishability of colors. Suppose that we have a solution using k colors. Then we see that this model has $\binom{|C|}{k}k!$ equivalent solutions. This property will be a great disadvantage when we use mixed integer linear programming (MILP) solvers. For this reason, cuts that remove the symmetry have been studied [3,4]. On the other hand, recently, alternative formulations for VCP have received a considerable attention. For instance, there are studies on a set partitioning formulation [5], an asymmetric representative formulation [6,7], an unconstrained quadratic binary programming formulation [8], and a supernodal formulation [9]. For further discussion on these formulations, see Burke et al. [9].

In this study, we focus on pairs of vertices which can be colored by the same color, and associate decision variables with these pairs. As a result, we obtain a new formulation for VCP. Our model does not suffer from the symmetry which is discussed above and has a linear fractional objective function. This objective function can be equivalently replaced by a piece-wise linear convex function, which gives us an MILP model for VCP. By this transformation, we can feed our model to commercial MILP solvers such as Gurobi Optimizer. To verify the validity of our formulation,

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we conducted numerical experiments on random graphs and several instances form DIMACS Implementation Challenge, and confirmed that it has a significantly good performance for dense graphs. It should be noted that high edge density does not necessarily make instances easy. In fact, there is a dense but hard instance `DSJC125.9` with only 125 vertices from DIMACS Implementation Challenge. The optimal value of this instance was an open problem until very recently. See Gualandi and Malucelli [10] for the details. We confirmed that our model solves this instance less than only 1 minute.

2. Our formulation

2.1. Expression as a fractional programming problem

In our formulation, for each distinct pair of vertices u and v, we introduce a decision variable x_{uv} which takes 1 if u and v share the same color and takes 0 otherwise. Clearly, we have $x_{uv} = 0$ for each $\{u, v\} \in E$. Here, we use the following inequality constraints

$$x_{uv} + x_{vw} - x_{uw} \leq 1 \quad (\forall u, v, w \in V \text{ with } u \neq v, v \neq w, u \neq w)$$

to obtain an explicit description of the feasible region. These inequalities say that if u and v share the same color ($x_{uv} = 1$) and v and w also share the same color ($x_{vw} = 1$), then u and v must receive the same color ($x_{uw} = 1$). These inequalities are referred to as the triangle inequalities studied in Grötschel and Wakabayashi [11] as facet-defining inequalities for a clique partitioning polytope. This relationship is natural because if $\boldsymbol{x}$ is a feasible solution for VCP, then a corresponding set $E_{\boldsymbol{x}} = \{\{u, v\} \mid x_{uv} = 1\}$ of edges induces a clique partitioning of the complement graph $\overline{G}$ of G, and vice versa.

Next, let us consider how to calculate the number of colors used in $\boldsymbol{x}$, namely the objective value. To this aim, we focus on the number of connected components in $(V, E_{\boldsymbol{x}})$, which equals the desired value. For a feasible solution $\boldsymbol{x}$, let us define

$$f_v(\boldsymbol{x}) = \frac{1}{1 + \sum_{u \in V} x_{uv}}$$

for each $v \in V$. Suppose that a vertex v belongs to a connected component (V', E') with $|V'| = k$ in $(V, E_{\boldsymbol{x}})$. Then we have $f_v(\boldsymbol{x}) = 1/k$ since V' is a clique of $(V, E_{\boldsymbol{x}})$. Thus, the sum of $f_v(\boldsymbol{x})$ over $v \in V'$ equals 1 for each connected component (V', E'), which implies that the sum of $f_v(\boldsymbol{x})$ over $v \in V$ gives the number of connected components in $(V, E_{\boldsymbol{x}})$. Therefore, we obtain the following proposition.

Proposition 1. *For a given feasible solution $\boldsymbol{x}$,*

$$\sum_{v \in V} f_v(\boldsymbol{x})$$

equals the number of connected components in $(V, E_{\boldsymbol{x}})$, which is the number of colors used in $\boldsymbol{x}$.

In sum, our formulation is described as follows:

$$\begin{aligned}
\text{minimize} \quad & \sum_{v \in V} f_v(\boldsymbol{x}) \\
\text{subject to} \quad & x_{uv} = 0 \quad (\forall \{u, v\} \in E), \\
& x_{uv} + x_{vw} - x_{uw} \leq 1 \\
& \quad (\forall u, v, w \in V \text{ with } u \neq v, v \neq w, u \neq w), \\
& x_{uv} \in \{0, 1\} \\
& \quad (\forall u, v \in V \text{ with } u \neq v).
\end{aligned}$$

It should be noted that there are redundant variables and constraints. Suppose that $\{u, v\} \in E$. Then, of course, we do not need to consider the decision variable x_{uv}. In addition, the transitivity constraints $x_{uv} + x_{vw} - x_{uw} \leq 1$ is redundant for every $w \in V \setminus \{u, v\}$ because it is equivalent to $x_{vw} - x_{uw} \leq 1$, which is satisfied by any pair of x_{vw} and x_{uw} with $0 \leq x_{vw}, x_{uw} \leq 1$. In our numerical experiments, such redundant variables and constraints are removed.

2.2. Expression as a mixed integer linear programming problem

To implement our formulation on MILP solvers, we propose to replace the fractional objective function by a piecewise linear convex function. For each $v \in V$, we introduce a continuous decision variable f_v which equals $f_v(\boldsymbol{x})$ for a given feasible solution $\boldsymbol{x}$. Namely, the objective function will be the sum of f_v over $v \in V$. For each $v \in V$ and for each $i \in \{0, 1, \ldots, n-1\}$, we add the following linear inequality constraint

$$f_v \geq u_i\left(\sum_{u \in V} x_{uv}\right), \tag{1}$$

where $u_i : \mathbb{R} \to \mathbb{R}$ is a linear function defined by

Table 1
Results of our formulation (ours for short) and the standard formulation (standard for short) for the randomly generated graphs.

Instance		Ours		Standard	
n	p	Time [s]	Gap [%]	Time [s]	Gap [%]
30	0.3	3.44	–	15.72	–
	0.5	4.48	–	122.91	–
	0.7	0.17	–	92.13	–
	0.9	0.04	–	34.34	–
50	0.3	*****	16.43	2018.34	–
	0.5	**	10.56	***	11.11
	0.7	11.71	–	*****	19.72
	0.9	0.14	–	*	4.00
70	0.3	*****	33.00	****	12.50
	0.5	*****	17.00	*****	30.73
	0.7	517.87	–	*****	27.46
	0.9	0.40	–	*****	12.41

Table 2
Results for the randomly generated graphs.

Instance		Ours	
n	p	Time [s]	Gap [%]
100	0.9	1.50	–
150	0.9	61.59	–
200	0.9	*	1.59

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