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On purely tree-colorable planar graphs

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ABSTRACT

A tree-k-coloring of a graph G is a k-coloring of G such that the subgraph induced by the union of any two color classes is a tree. G is purely tree-k-colorable if the chromatic number of G is k and any k-coloring of G is a tree-k-coloring. Xu [16] conjectured that there exist only two purely tree-4-colorable 4-connected maximal planar graphs. In this paper, we construct an infinite family of purely tree-colorable 4-connected maximal planar graphs, called dumbbell-maximal planar graphs, which disprove Xu's conjecture. Moreover, we give the enumeration of dumbbell-maximal planar graphs and propose a conjecture on such graphs. It turns out that the conjecture implies naturally the uniquely 4-colorable planar graph conjecture.

1. Introduction

All graphs considered in this paper are finite, simple and undirected, and we follow [1] for the terminologies and notations not defined here. Given a graph G, we use $V(G)$, $E(G)$ and $\delta(G)$ (or simply V, E and δ if the graph is clear from the context) to denote the *vertex set*, the *edge set* and the *minimum degree* of G, respectively. A subgraph H of G is called an *induced subgraph* if for any $u, v \in V(H)$, u, v are adjacent in G if and only if they are adjacent in H; we also say H is a subgraph *induced* by $V(H)$ in the traditional sense, written as $H = G[V(H)]$. A *k-path* (or *k-cycle*) is a path (or cycle) of length k. An *n-wheel* is a graph on $n+1$ vertices, which is constructed by an n-cycle and a more vertex adjacent to each vertex of the cycle.

A *planar graph* is a graph that can be drawn in the plane so that its edges intersect only at their ends. A graph is called a *maximal planar graph* (MPG) or a *triangulation* if it is planar but adding any edge (on the given vertex set) would destroy that property. If an MPG can be reduced into the tetrahedral graph by deleting a 3-vertex and its incident edges, repeatedly, then we call this graph a *recursive MPG*, where a k-vertex of a graph G is a vertex with degree k. A cycle C of a planar graph is *separating* if there exist vertices in the interior and the exterior of C.

A *k-coloring* of G is an assignment of k colors to $V(G)$ such that no two adjacent vertices are assigned the same color. Naturally, a k-coloring can be viewed as a partition $\{V_1, V_2, \cdots, V_k\}$ of V, where V_i denotes the set of vertices assigned color i, and is called a *color class* of the coloring for any $i = 1, 2, \cdots, k$. A graph G is *k-colorable* if it admits a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G is k-colorable. A graph G is *uniquely k-colorable* if $\chi(G) = k$ and G has only one k-coloring up to permutation of the colors.

The uniquely coloring problem of graphs was first proposed by Cartwright and Harary [2] and Gleason and Cartwright [8]. In 1973, Greenwell and Kronk [11] studied the uniquely colorable graphs in terms of the edge coloring, and proposed a conjecture as follows.

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Conjecture 1.1. *If G is a uniquely 3-edge-colorable cubic graph, then G is a planar graph that contains a triangle.*

In 1975, Fiorini [3] independently studied uniquely edge colorable graphs, and obtained some similar results to the ones of Greenwell and Kronk. After that, many scholars discussed this class of graphs, such as Thomason [14,15], Fiorini and Wilson [4,5], Zhang [18], and Goldwasser and Zhang [9,10]. In 1977, Fiorini and Wilson [4] put forward the following conjecture on the basis of Conjecture 1.1.

Conjecture 1.2 *(Uniquely 4-colorable planar graph conjecture: edge version). Every uniquely 3-edge-colorable cubic planar graph contains a triangle.*

Fisk [6] independently proposed a dual version of Conjecture 1.2, which characterizes the structure of uniquely 4-colorable planar graphs.

Conjecture 1.3 *(Uniquely 4-colorable planar graph conjecture: vertex version). A planar graph G is uniquely 4-colorable if and only if G can be obtained from K_4 by embedding a vertex of degree 3 in some triangular face continuously, that is, G is a recursive MPG.*

Goldwasser and Zhang [10] proved that every counterexample to Conjecture 1.3 is 5-connected. Fowler [7] investigated in detail the uniquely 4-colorable planar graphs by using a method similar to the proof of the 4-Color Theorem [13].

For a k-coloring f of a graph G, if the subgraph induced by the union of any two color classes under f is a tree, then we call f a *tree-k-coloring* of G. If the chromatic number of G is k and any k-coloring of G is a tree-k-coloring, then G is called *purely tree-k-colorable*. Note that a tree-k-coloring is also an acyclic k-coloring, which was introduced by Grunbaum [17].

By definition, it can be seen that each purely tree-k-colorable graph is connected. Moreover, we have the following Lemma 1.4, which is straightforward to prove.

Lemma 1.4. *If G is a purely tree-k-colorable graph on n vertices, then*

$$|E(G)| = \frac{1}{2}(k-1)(2n-k).$$

Conversely, if G is uniquely k-colorable and $|E(G)| = \frac{1}{2}(k-1)(2n-k)$, then for any k-coloring of G, the subgraph induced by the union of any two color classes is a tree. So we can obtain the following theorem.

Theorem 1.5. *If G is uniquely k-colorable and $|E(G)| = \frac{1}{2}(k-1)(2n-k)$, then G is purely tree-k-colorable.*

In this paper, we mainly consider the purely tree-4-colorable planar graphs. It is well known that the maximum number of edges in a planar graph with $n \geq 3$ is $3n-6$, in which case the planar graph is maximal. Using this fact, together with Lemma 1.4, we can conclude the following result.

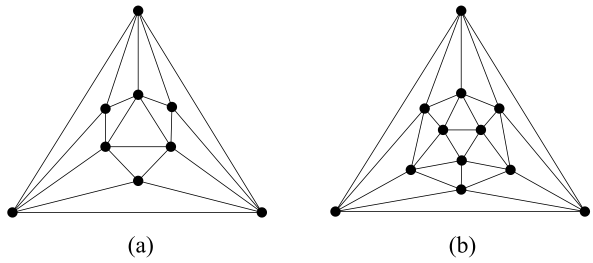

Fig. 1. Two purely tree-4-colorable MPGs J^9 and I^{12}.

Corollary 1.6. *Every purely tree-4-colorable planar graph is a maximal planar graph.*

In 2005, Xu [16] found two purely tree-4-colorable 4-connected maximal planar graphs (MPGs) J^9 and I^{12} (icosahedron) shown in Figs. 1(a) and (b), and conjectured that there does not exist any purely tree-4-colorable MPG except for J^9 and I^{12}. In this paper, we construct an infinite family of purely tree-4-colorable 4-connected MPGs, called dumbbell-maximal planar graphs (dumbbell-MPGs), which disprove Xu's conjecture. Moreover, we conjecture that a 4-connected MPG G is purely tree-4-colorable if and only if G is either the icosahedron or a dumbbell-MPG, which implies naturally the uniquely 4-colorable planar graph conjecture.

2. Purely tree-4-colorable planar graphs

2.1. Construction

A *dumbbell* is a graph consisting of two triangles $\triangle v_1v_2u$ and $\triangle uv_3v_4$ with exactly one common vertex u (see Fig. 2(a)). Obviously, a 4-wheel contains exactly two dumbbells. Without special assertion, dumbbells considered in this paper are ones contained in a 4-wheel.

The *dumbbell transformation* is defined as follows. For a given dumbbell $X = \triangle v_1v_2u \bigcup \triangle uv_3v_4$, first, add two 3-vertices x_1 and x_2 on the two triangular faces of X, respectively. Then split the vertex u into two vertices u and u', and split the edges xu and uy into two edges xu, xu' and $uy, u'y$ respectively. Hence, the vertices x, u', y, u form a 4-cycle. Then add a new vertex v in this cycle adjacent to every vertex of the cycle. The process is shown in Figs. 2(a)–(c).

It is easy to prove the following theorem.

Theorem 2.1. *Let G be an MPG with a 4-wheel W_4. Then the graphs obtained from G by implementing the dumbbell transformations on two dumbbells of W_4 are isomorphic.*

A graph G is a *dumbbell-MPG* if either G is isomorphic to J^9, or G can be obtained from a dumbbell-MPG by implementing a dumbbell transformation. We denote by J^n a dumbbell-MPG on n vertices. For instance, Fig. 3(c) is a dumbbell-MPG on 13 vertices, which can be obtained