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A strengthened analysis of a local algorithm for the minimum dominating set problem in planar graphs

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ABSTRACT

In recent years growing interest in local distributed algorithms has widely been observed. This results from their high resistance to errors and damage, as well as from their good performance, which is independent of the size of the network. A local deterministic distributed algorithm finding an approximation of a Minimum Dominating Set in planar graphs has been presented by Lenzen et al., and they proved that the algorithm returns a 130-approximation of the Minimum Dominating Set. In this article we will show that the algorithm is two times more effective than was previously assumed, and we prove that the algorithm by Lenzen et al. outputs a 52-approximation to a Minimum Dominating Set. Therefore the gap between the lower bound and the approximation ratio of the best yet local deterministic distributed algorithm is reduced by half.

1. Introduction

A distributed algorithm is called a local algorithm if it runs in a constant number of synchronous communication rounds. In recent years growing interest in local distributed algorithms has widely been observed. This results from their high resistance to errors and damage, as well as from their good performance, which is independent of the size of the network. These properties allow for them to be used in practice. Recently many different algorithms have been proposed which solve such optimisation problems as Minimum Dominating Set, Minimum Edge Cover or Semi-matching. However, it appears that an exact solution to optimisation problems is frequently impossible to achieve in a reasonable amount of time (e.g. NP-complete problems), which is the reason why algorithms finding approximate solutions to such problems are taken into consideration.

Research on local distributed algorithms has been conducted for several decades now (see book [6]) and numerous papers have been published. Perhaps the best way to explore the subject more thoroughly is to read an excellent work by Suomela entitled "Survey of Local Algorithms" [7], in which the author describes all of the major research results in the field of local algorithms.

A *dominating set* in a graph $G = (V, E)$ is a subset of vertices $D \subseteq V$ such that every vertex $v \in V$ is an element of D or is adjacent to at least one vertex from D. A dominating set of the smallest possible size in graph G is called a *Minimum Dominating Set* and will be denoted by M.

Because finding a Minimum Dominating Set is an NP-complete problem even in planar graphs, we focus on the constant approximation of this problem in planar graphs. It is also known that there is an algorithm which finds an $(1+\varepsilon)$-approximation of the Minimum Dominating Set in this class of graphs in $O(\log^* n)$ rounds [1]. The first correct local algorithm for planar graphs was given in [4] and shown to yield a 130-approximation of this problem. This result is especially significant, because most of the previous local algorithms work on graphs of bounded degree. The fact that such an algorithm exists is somewhat surprising, given that Czygrinow et al. in [1] and Lenzen, Wattenhofer in [5] proved that there are no local constant factor algorithms for the problem of finding a maximum independent set, nor is there maximum

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matching in planar graphs. Czygrinow et al. also showed that there does not exist any local algorithm that can find a $(5-\varepsilon)$-approximation of the Minimum Dominating Set problem in planar graphs. A reduction of the gap between the existing lower bound and the current algorithm is an interesting aspect that is raised in this article.

Recently it has been shown in [8] that there actually exists an algorithm finding a 636-approximation of this problem in planar graphs in a model without unique identifiers. The issue of the resemblance of models with and without identifiers has been raised in [2,4].

1.1. Model and notation

In this paper we work in a synchronous communication model and we use a planar graph $G = (V_G, E_G)$ as a representation of the network. The edges in the graph will correspond to communication links and vertices from the set V_G will correspond to the processors.

In this model, algorithms are executed in synchronous communication rounds. In each round every processor is able to send and receive messages from/to all of its neighbours and to perform local computations based on information that has been gathered so far. We assume that each vertex has a unique identifier of length $O(\log n)$ bits, where n is the size of the network. In order to facilitate the reader to understand this paper, we use similar notations as in paper [4]. For vertices $A \subseteq V_G$ we define inclusive neighbourhood of A in graph G as $N_A^+(G) := \{v \in V_G\colon\ v \in A \vee \exists e = uv \in E_G\colon\ u \in A\}$. We also denote the neighbours of A that are not in A by $N_A(G) := N_A^+(G) \setminus A$. To simplify the notation in cases in which $A = \{a\}$, we may omit the braces, e.g. $N_a(G)$ instead of $N_{\{a\}}(G)$. For $v \in V_G$ we also define a set of nodes with a distance at most two to the vertex v as $N_v^{(2)}(G) := N_X^+(G)$, where $X := N_v^+(G)$.

We also need to introduce some graph theoretical terminology for planar graphs, which are necessary only in an analysis of the procedure. For a plane graph G in $\mathbb{R}^2$ we define a face of G as a maximal open set f in $\mathbb{R}^2 \setminus G$ such that any two points in f can be connected by a curve contained in f. There is exactly one unbounded face f_u, and the other faces are called inner faces. Furthermore, let $\mathrm{Reg}[G]$ be the set of points $\mathbb{R}^2$ without unbounded face f_u ($\mathrm{Reg}[G] := \mathbb{R}^2 \setminus f_u$). We assume that C is the smallest boundary walk of G containing all edges adjacent to face f_u. Then we set $\mathrm{Reg}(G) := \mathrm{Reg}[G] \setminus C$. If a vertex $v \in V_{G'}$ is contained in the set of points $\mathrm{Reg}[G]$ (or $\mathrm{Reg}(G)$), where G is a subgraph of G' then we write $v \in \mathrm{Reg}[G]$ (or $v \in \mathrm{Reg}(G)$). Moreover, if each vertex and each edge from G is contained in the set of points $\mathrm{Reg}[G']$ (or $\mathrm{Reg}(G')$), we write $G \subseteq \mathrm{Reg}[G']$ (or $G \subseteq \mathrm{Reg}(G')$). By $V_{G'} \cap \mathrm{Reg}[G]$ we denote all vertices of $V_{G'}$ lying inside the set of points $\mathrm{Reg}[G]$.

2. Analysis of the algorithm from paper [4]

In this section an improved analysis of the algorithm from paper [4] will be presented. Lenzen, Oswald and Wattenhofer proposed an algorithm that finds a 130-approximation of a Minimum Dominating Set in planar graphs.

Algorithm 1. MDS approximation in planar graphs.

$D_1 := \emptyset$, $D_2 := \emptyset$.
for $v \in V_G$ in parallel **do**
    if $\nexists A \subseteq N_v^{(2)}(G) \setminus \{v\}$ such that $N_v(G) \subseteq N_A^+(G)$ and $|A| \leqslant 6$ **then**
        $D_1 := D_1 \cup \{v\}$
    end if
end for
for $v \in V_G$ in parallel **do**
    $\bar{\delta}_v := |N_v^+(G) \setminus N_{D_1}^+(G)|$ ▹ residual degree
    if $v \in V_G \setminus N_{D_1}^+(G)$ **then**
        $\Delta_v := \max_{w \in N_v^+(G)} \{\bar{\delta}_w\}$ ▹ maximum within one hop
        choose any $d(v) \in \{w \in N_v^+(G) | \bar{\delta}_w = \Delta_v\}$
        $D_2 := D_2 \cup \{d(v)\}$
    end if
end for
return $D_1 \cup D_2$.

We show that, as a matter of fact, the value of this factor is more than two times better than was assumed, and that it is a 52-approximation of an optimal solution. First we will recall the formal pseudocode of the algorithm and how it functions.

Execution of the algorithm is divided into two phases. The first phase is responsible for adding, to the dominating set D, vertices whose neighbourhoods cannot be dominated by a small number of other vertices. It has been proved that the size of such a set is less than three times bigger than the order of an optimal solution M. In the second phase each vertex that is not yet dominated adds one vertex to D, either itself or one of its neighbours. It is always a vertex, from inclusive neighbourhood, dominating the highest number of not yet dominated vertices that is chosen.

2.1. Analysis of the algorithm

We note that, although, the only assumption in the algorithm is that G is planar, an analysis of the procedure uses properties of a plane drawing of G. Let M be an optimal solution in plane graph $G = (V_G, E_G)$, and D_1 and D_2 as in the algorithm above. In paper [4] the following bounds have been proved.

Lemma 1 *(Lemma 4.2 in [4]).* $|D_1 \setminus M| < 3|M|$.

Lemma 2 *(Conclusion in [4]).* $|D_2| < 126|M|$.

Notice that there is a large disproportion between the upper bounds of sets D_1 and D_2. An improvement of the upper bound of set D_2 turns out to be a key issue in the new analysis of the algorithm. Although it will be based, just as in [4], on a division of the graph into special subgraphs (see the double star in Fig. 1), it is different and does not require taking into consideration as many cases as in the original work.

Definition 1. Let $H = (V_H, E_H)$ be a subgraph of a planar graph $G = (V_G, E_G)$ obtained in the following way:

1. Let $V_H := \emptyset$, $E_H := \emptyset$.
2. For each vertex $d \in D_2 \setminus M$, add to the set V_H one vertex $v \in V_G \setminus M$ such that $d = d(v)$ (if possible). Let A denote the set of all added vertices.