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Maximum common induced subgraph parameterized by vertex cover

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A R T I C L E I N F O

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A B S T R A C T

The Maximum Common Induced Subgraph problem (*MCIS*) takes a pair of graphs as input and asks for a graph of maximum order that is isomorphic to an induced subgraph of each of the input graphs. The problem is $\mathcal{NP}$-hard in general, and remains so on many graph classes including graphs of bounded treewidth. In the framework of parameterized complexity, the latter assertion means that *MCIS* is $W[1]$-hard when parameterized by the treewidths of input graphs.

A classical graph parameter that has been used recently in many parameterization problems is Vertex Cover. In this paper we prove constructively that *MCIS* is fixed-parameter tractable when parameterized by the vertex cover numbers of the input graphs. Our algorithm is also an improved exact algorithm for the problem on instances where the minimum vertex cover is small compared to the order of input graphs.

1. Introduction

A common induced subgraph of two graphs G_1 and G_2 is a graph H that is isomorphic to induced subgraphs of both G_1 and G_2. The *Maximum Common Induced Subgraph* problem, henceforth *MCIS*, takes a pair of graphs as input and asks for a common induced subgraph, of maximum order, of the two graphs. Another well studied version of *MCIS* seeks an induced subgraph with the maximum number of edges [5], and in some applications the target subgraph is also required to be connected [15].

MCIS has been studied intensively due to its applications in a number of domains, including Bioinformatics [13,16,25], Chemistry [20,22] and Pattern Recognition [7,18]. The problem is $\mathcal{NP}$-hard, by a simple reduction from Maximum Clique, which corresponds to the special case where G_1 and G_2 have the same order and G_1 is complete. A similar reduction proves the $W[1]$-hardness of the problem, when parameterized by the solution size: Let H be a fixed clique of order k. For each input G of k-Clique, the reduction simply computes the instance (G, H) of k-*MCIS*.

We also observe that *MCIS* remains $W[1]$-hard when restricted to bipartite graphs. This follows from the $W[1]$-hardness of the Induced Matching problem [21]: just assume G_1 is a disjoint union of k edges. We assume familiarity with Parameterized Complexity Theory and Fixed-Parameter Algorithm. We refer the readers to [10] for more details about this topic.

Structural parameterization is of fundamental importance in studying the parameterized complexity of $\mathcal{NP}$-hard problems. Many problems that are $W[1]$-hard when parameterized by natural parameters, such as solution size or size of a partition, become fixed-parameter tractable when auxiliary parameters are used. Treewidth is undoubtedly the most commonly used auxiliary problem parameter since the 1990s. Recently, it was shown that Common *Connected* Subgraph Isomorphism is not fixed-parameter tractable when parameterized by treewidth [3]. We note that *MCIS* is also W[1]-hard in this case. This can be inferred easily from the fact that Subgraph Isomorphism is $\mathcal{NP}$-hard on graphs of bounded treewidth [19].

More recently, the size of a minimum vertex cover was used as auxiliary parameter in a number of graph problems [9,11,12]. This, along with the above negative results, prompted us to study the parameterized complexity of

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MCIS when the parameter k is a bound on the minimum vertex covers of the input graphs. We show, constructively, that *MCIS* is fixed-parameter tractable in this case.

2. Background

Throughout this paper we only consider graphs that are simple and undirected, and we adopt common graph theoretic terminology. For a set S of vertices of a graph G, $N(S)$ denotes the set of neighbors of the elements of S that are not in S. A vertex cover of a graph G is a set of vertices whose complement induces an edgeless subgraph. A k-vertex cover is a vertex cover of cardinality k or less.

In the process of searching for a largest common induced subgraph of a pair (G_1, G_2) of graphs, we seek a mapping, f, from $V(G_1)$ to $V(G_2) \cup \{none\}$ where *none* is the image of vertices that are not part of the common induced subgraph and f is an isomorphism when restricted to $V(G_1) \setminus f^{-1}(\{none\})$. In other words, for each pair $u, v \in V(G_1) \setminus f^{-1}(\{none\})$, $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. We shall refer to f as *MCIS-mapping*. In this context, if $v = f(u)$, then we say u and v are matched (or form a matched pair). We also say that u is matched with v, and vice versa.

Two pairs (u, v) and (u', v') of $V(G_1) \times V(G_2)$ are said to be compatible if the following holds: u and u' are adjacent in G_1 if and only if v and v' are adjacent in G_2. We shall sometimes refer to the *MCIS*-mapping f as a set of compatible pairs that are matched under f. Moreover, a pair (u, v) is *compatible with* f if it is compatible with all pairs in f.

During the search for a common induced subgraph, we often build an auxiliary bipartite graph $H(f) = (A, B)$, called a *compatibility graph* in this paper. H may be updated dynamically as we match/unmatch pairs of vertices of the two graphs. The vertex sets A and B of H are subsets of $V(G_1)$ and $V(G_2)$, respectively. Edges of H join vertices that can still be matched, given previously matched vertices. The non-isolated vertices of H are said to be *active* during the search process.

Remark 1. Every *MCIS*-mapping yields a matching in the compatibility graph H such that pairs of matching edges correspond to compatible pairs. In the special case where the two vertex sets of H correspond to independent sets of G_1 and G_2, *MCIS* reduces to finding a maximum matching in H.

The asymptotically-fastest exact algorithm for *MCIS* is based on mere brute-force enumeration of all the induced subgraphs of the given graphs, and in each case applying the best-known algorithm for Graph Isomorphism. This simple approach runs in

$$\mathcal{O}\big(2^{(n_1+n_2)\sqrt{\mathcal{O}(\min(n_1,n_2)\log(\min(n_1,n_2)))}}\big)$$

[3], where $n_i = |V(G_i)|$. Another approach that attempts at improving the worst-case scenario was presented in [2] where the vertex cover of one of the input graphs is assumed to be small. In fact, fixing the size of a minimum vertex cover in one of the input graphs does not make the problem simpler, from a parameterized complexity standpoint, as observed in the following.

Lemma 2. *The Maximum Common Induced Subgraph problem, parameterized by the size of a minimum vertex cover of only one of the input graphs, remains $W[1]$-hard even if the input is restricted to bipartite graphs.*

Lemma 2 can be easily proved by reduction from the $W[1]$-hard Bipartite Induced Matching problem: again, for an arbitrary input (G, k) of Bipartite Induced Matching, let G_1 be a disjoint union of k edges and let $G_2 = G$.

We shall prove, constructively, that *MCIS* is fixed-parameter tractable when (fully) parameterized by the vertex cover number. In other words, we present an exact algorithm whose running time is exponential in the size of the vertex covers of the input graphs.

3. Maximum common subgraph parameterized by vertex cover

We now consider instances of *MCIS* where the parameter k is a bound on the size of minimum vertex covers of the input graphs. In other words, *MCIS* is posed as an optimization problem on graphs whose vertex cover does not exceed k. We refer to this problem by *MCIS_VC* in the sequel.

Let (G_1, G_2, k) be an instance of *MCIS_VC* and let C_1 and C_2 be k-vertex covers of G_1 and G_2, respectively. Denote by I_j the complement of C_j in $V(G_j)$ and let $n = \min(|V(G_1)|, |V(G_2)|)$. We also denote by C_{j1} and C_{j2} the sets of elements of C_j that are matched, under some given *MCIS*-mapping f, with elements of C_{3-j} and I_{3-j}, respectively $(j \in \{1, 2\})$.

We first observe that a common induced subgraph of size $n - k$ is easily found: just take the two isomorphic (edge-less) subgraphs induced by I_1 and I_2. To find an optimum solution, our algorithm seeks a common subgraph of size $> n - k$. The following key lemma is used to obtain such solution, if it exists.

Lemma 3. *Let f be an optimum MCIS-mapping of the instance (G_1, G_2, k) of MCIS_VC, and let C_{j1} and C_{j2} be as defined above. If the input graphs have a common induced subgraph whose order exceeds $n - k$, then $|N(C_{j2})| < 2k$.*

Proof. Let $u \in C_{12}$ and $v \in I_2$ be such that $v = f(u)$, and let $u' \in N(u)$. If u' is matched with v' in G_2, then $v' \notin I_2$, otherwise the pair (u, v) and (u', v') would be incompatible with respect to f. It follows that elements of $N(C_{12})$ can only be matched with elements of the k-vertex cover of the other graph. The same applies to $N(C_{22})$. Accordingly, if $|N(C_{j2})| \geqslant 2k$, then at least k elements of $N(C_{j2})$ are unmatched under f, which makes the common edgeless subgraph of order $n - k$ a maximum solution. □

Our algorithm starts by branching on the elements of the vertex cover C_1 of G_1. Instead of trying to match a vertex of C_1 with any of the $|V(G_2)|$ vertices of G_2, and to make use of Lemma 3, we distinguish elements of C_1 that are matched with vertices of I_2 as follows: