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Triangle strings: Structures for augmentation of vertex-disjoint triangle sets ☆

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ABSTRACT

Vertex-disjoint triangle sets (triangle sets for short) have been studied extensively. Many theoretical and computational results have been obtained. While the maximum triangle set problem can be viewed as the generalization of the maximum matching problem, there seems to be no parallel result to Berge's augmenting path characterization on maximum matching (C. Berge, 1957 [1]). In this paper, we describe a class of structures called triangle string, which turns out to be equivalent to the class of union of two triangle sets in a graph. Based on the concept of triangle string, a sufficient and necessary condition that a triangle set can be augmented is given. Furthermore, we provide an algorithm to determine whether a graph G with maximum degree 4 is a triangle string, and if G is a triangle string, we compute a maximum triangle set of it. Finally, we give a sufficient and necessary condition for a triangle string to have a triangle factor.

1. Introduction, definitions and terminologies

We consider undirected, simple graphs in this paper. Let G be a graph. A set $\mathcal{T}$ of vertex disjoint triangles in G is called a *vertex-disjoint triangle set* of G. For short, we call $\mathcal{T}$ a *triangle set* of G in this paper. The number of triangles in $\mathcal{T}$, denoted by $|\mathcal{T}|$, is called the *size* of it. A triangle set of G with the maximum size is called a *maximum triangle set* of G. We say that a vertex u is *covered* by a triangle set $\mathcal{T}$, if u is a vertex of a triangle in $\mathcal{T}$. If $\mathcal{T}$ covers all vertices of G, we say that $\mathcal{T}$ is a *triangle factor* of G.

The study on triangle sets and triangle factors has a long history. Important results include sufficient conditions for the existence of triangle factors in graphs, and bounds on the size of the maximum triangle sets in graphs. For example, the following fundamental result is a special case of a theorem in [5].

Theorem 1.1. *(See Corrádi and Hajnal [5], 1963.) If G is a graph with $3k$ vertices and minimum degree of at least $2k$ then G contains a triangle factor.*

While in tripartite graphs the bound can be reduced.

Theorem 1.2. *(See Johansson [8], 2000.) Let G be a tripartite graph with $3k$ vertices, k in each class, such that each vertex is connected to at least $\frac{2}{3}k+\sqrt{k}$ of the vertices in each of the other two classes, then G has a triangle factor.*

Another example is a result of the size of triangle sets in claw-free graphs.

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Theorem 1.3. *(See Wang [10], 1998.) For any integer $k \geqslant 2$, if G is a claw-free graph of order at least $6(k-1)$ and with minimum degree at least 3, then G contains a triangle set of size k unless G is of order $6(k-1)$ and G belongs to a known class of graphs.*

The problem of computing the maximum triangle sets (called the vertex-disjoint triangles problem, VDT for short) in graphs catches much attention. The VDT problem has many variants such as computing the maximum triangle sets in edge-weighted graphs [7], in degree-bounded graphs ([2–4] and [9]), or in some special classes of graphs [6]. Particularly, in [2], Caprara and Rizzi prove that the VDT problem is APX-hard for graphs with maximum degree 4.

Triangle sets can be viewed as a generalization of matchings in graphs. For matching problems, Berge's famous characterization says that a matching M in a graph G is maximum if and only if G has no M-augmenting path [1]. However, for triangle sets in graphs, there seems no similar augmenting results. In this paper, we describe a class of structures called triangle string, which corresponds to the union of the graphs of two triangle sets. Based on the concept of triangle string, we give a sufficient and necessary condition under which a triangle set $\mathcal{T}$ of a graph G can be augmented. We describe an algorithm which determines whether a given graph G with degree bound 4 is a triangle string, and if G is a triangle string, find out a maximum triangle set of it. Finally we give a sufficient and necessary condition under which a triangle string has a triangle factor.

We use $\langle\mathcal{T}\rangle$ to denote the graph consisting of all the triangles in a triangle set $\mathcal{T}$, and say that it is the *graph* of $\mathcal{T}$. We often consider $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$, the union of the graphs of two triangle sets $\mathcal{T}_1$ and $\mathcal{T}_2$, which is called the *union graph* of $\mathcal{T}_1$ and $\mathcal{T}_2$ in this paper.

Let u be a vertex of degree d in a graph G. Then we say that u is a *d-vertex* in G. Let $T = uvwu$ be a triangle in G, where the degree of u, v and w are d_u, d_v and d_w in G. Then we say that T is a *(d_u, d_v, d_w)-triangle* in G. We also say that the *degree sequence* of T in G is (d_u, d_v, d_w).

2. Triangle strings

In this section we describe the triangle string structure. The following operations are to be used in the construction of triangle strings.

Let G be a graph. The operations (1), (2) and (3) below are called *triangle-additions*.

(1) Let w be a 2-vertex in G, add two new vertices u and v to G, and add edges to form a triangle on u, v and w.
(2) Let v and w be two nonadjacent 2-vertices at odd distance in G, add a new vertex u to G, and add edges to form a triangle on u, v and w.
(3) Let u, v and w be three pairwise nonadjacent 2-vertices at pairwise odd distance in G, add edges to form a triangle on u, v and w.

Denote by K_4^- the graph obtained from K_4 by removing an edge. An *hourglass* is a graph isomorphic to $K_5 - E(C_4)$, i.e. the graph obtained by removing the edges of a 4-cycle from a K_5. The following operations (4) and (5) are called *K_4^--insertions*.

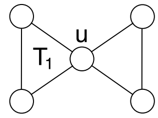

Fig. 1. Case (2): One vertex of T_1 covered by $\mathcal{T}_2$.

(4) Let w be a 2-vertex in G and u, v the neighbors of w. Replace w with a K_4^- denoted by K, and let u, v be adjacent to one 2-vertex w_1 in K.
(5) Let w be the 4-vertex in an induced hourglass in G. Let the neighbors of w be u_1, u_2, v_1 and v_2, where $u_1v_1 \in E(G)$ and $u_2v_2 \in E(G)$. Replace w with a K_4^- denoted by K, and let u_1 and v_1 be adjacent to a 2-vertex w_1 in K, u_2 and v_2 be adjacent to another 2-vertex w_2 in K.

We call a K_4^- in a graph G connected to the other parts of G through the 2-vertices in it a *swing* K_4^-. Hence, operation (4) and (5) insert swing K_4^-'s into G.

A *vertex-jointed triangle string* S is either a triangle, or obtained from another vertex-jointed triangle string S' by performing one triangle addition.

A *triangle string* S is either a K_4^-, or a vertex-jointed triangle string, or obtained from another triangle string S' by performing a K_4^--insertion.

It is not hard to see the following properties of a triangle string S.

(a) Every vertex in S should be of degree 2, 3 or 4.
(b) S is vertex-jointed if and only if it contains no vertex of degree 3.
(c) Every 2-vertex in S is contained in one triangle.
(d) Every 3-vertex in S is a 3-vertex in a swing K_4^-, and hence it is contained in two triangles.
(e) Every 4-vertex in S is the 4-vertex in an induced hourglass, and hence it is contained in two triangles. Furthermore, operation (5) is valid for every 4-vertex in S.
(f) A triangle in S can only have one of the following six degree sequences in S: $(2, 2, 2)$, $(2, 2, 4)$, $(2, 4, 4)$, $(4, 4, 4)$, $(2, 3, 3)$ and $(3, 3, 4)$.

3. Union graph of two triangle sets and an augmenting theorem

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangle sets in a graph G, and $T_1 = uvwu$ be a triangle in $\mathcal{T}_1$. Considering how the vertices of T_1 be covered by the triangles in $\mathcal{T}_2$, we have the following cases on the structure near T_1 and the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$.

(1) All vertices of T_1 are not covered by $\mathcal{T}_2$. Then T_1 forms a component of $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$, and the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$ is $(2, 2, 2)$.
(2) There is one vertex u of T_1 covered by $\mathcal{T}_2$, as shown in Fig. 1. Then, u is a vertex of degree 4 in $\langle\mathcal{T}_1\rangle \cup$